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SOME PROBLEMS IN THE ANALYTIC THEORY OF NUMBERS

By E. C. TITCHMARSH (*Oxford*)

[Received 26 July 1942]

1. In Hardy's 'circle method' of solving problems in the theory of numbers, an arithmetic expression is made to depend on an integral involving an analytic function which has singularities all round the unit circle. It is shown that the integral is, in a sense, the sum of the contributions of the separate singularities, and the sum of the series so formed provides an asymptotic formula for the arithmetic expression.

This method has been applied by Estermann* to the function

$$f_2(z) = \sum_{n=1}^{\infty} d(n)z^n, \tag{1.1}$$

where $d(n)$ is the number of divisors of n. In this way asymptotic formulae for the sums

$$\sum_{k+l=n} d(k)\,d(l), \qquad \sum_{k+l+m=n} d(k)\,d(l)\,d(m)$$

were obtained. A still simpler application would give the results

$$\sum_{n=1}^{\infty} d^2(n)e^{-2n\delta} \sim \frac{1}{2\pi^2\delta}\log^3\frac{1}{\delta} \quad (\delta \to 0) \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} d(n)\,d(n+r)e^{-2n\delta} \sim \frac{3}{\pi^2}\frac{\sigma(r)}{r}\frac{1}{\delta}\log^2\frac{1}{\delta}, \tag{1.3}$$

where r is a fixed positive integer, and $\sigma(r)$ is the sum of the divisors of r. It would, however, not be worth while carrying out the analysis for (1.2) and (1.3) in detail, since they can be proved more easily in other ways.†

I recently attempted to see what could be done in a similar way with the function

$$f_3(z) = \sum_{n=1}^{\infty} d_3(n)\,z^n, \tag{1.4}$$

* T. Estermann, 'On the representations of a number as the sum of three products': *Proc. London Math. Soc.* (2) 29 (1929), 453–78; and 'On the representations of a number as the sum of two products': ibid. 31 (1930), 123–33.

† See B. M. Wilson, 'Proofs of some formulae enunciated by Ramanujan': *Proc. London Math. Soc.* (2) 21 (1922), 235–55; A. E. Ingham, 'Some asymptotic formulae in the theory of numbers': *J. of London Math. Soc.* 2 (1927), 202–5; T. Estermann, 'Über die Darstellungen einer Zahl als Differenz von zwei Produkten': *J. für Math.* 164 (1931), 173–82.

where $d_3(n)$ is the number of ways in which n can be expressed as the product of three factors. Here grave difficulties are to be expected with the error terms. Still, it might be expected that plausible results could be obtained in a number of problems by proceeding in the same way as with $f_2(z)$, even if rigorous proofs were not at present available.

The main result of the present paper is that this conjecture is false. It can be proved in another way that

$$\sum_{n=1}^{\infty} d_3^2(n) e^{-2n\delta} \sim \frac{A_1}{2.8!} \frac{1}{\delta} \log^8 \frac{1}{\delta}, \tag{1.5}$$

where*

$$A_1 = \prod_p \left(1-\frac{1}{p}\right)^4 \left(1+\frac{4}{p}+\frac{1}{p^2}\right). \tag{1.6}$$

However, if we apply the circle method to (1.5), we obtain, not the above right-hand side, but

$$\frac{17A_1}{2^{16}.21} \frac{1}{\delta} \log^8 \frac{1}{\delta}.$$

This is $\frac{255}{256}$ of the correct value. It seems curious that the result should be so nearly right without being quite right. In the absence of any method of dealing with what, by analogy with $f_2(z)$, one would suppose to be error terms, no explanation of this state of affairs can be given.

The matter is still further complicated by the fact that the result

$$\sum_{n=1}^{\infty} d(n)\, d_3(n) e^{-2n\delta} \sim \frac{A_2}{240} \frac{1}{\delta} \log^5 \frac{1}{\delta}, \tag{1.7}$$

where

$$A_2 = \prod_p \left(1-\frac{1}{p}\right)^2 \left(1+\frac{2}{p}\right), \tag{1.8}$$

is given correctly by applying the circle method to $f_2(z)$ and $f_3(z)$ together.

The circle method also gives the results

$$\sum_{n=1}^{\infty} d_3(n)\, d_3(n+r) e^{-2n\delta} \sim K_1(r) \frac{1}{\delta} \log^4 \frac{1}{\delta} \tag{1.9}$$

and

$$\sum_{n=1}^{\infty} d_3(n)\, d(n+r) e^{-2n\delta} \sim K_2(r) \frac{1}{\delta} \log^3 \frac{1}{\delta}, \tag{1.10}$$

where $K_1(r)$, $K_2(r)$ are constants depending on r. No alternative

* A_1 and A_2 denote the same constants throughout the paper.

method of obtaining these formulae appears to be known. Probably (1.10) is true, since we succeed with (1.7), but (1.9) is more doubtful.

2. We shall use the following lemmas.

LEMMA α. *Let* $$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (\sigma > 1),$$
where $a_n = O(n^\epsilon)$ *for every positive* ϵ; *and let*
$$f(s) = \zeta^k(s) g(s),$$
where k *is a positive integer, and* $g(s)$ *is regular and bounded in a half-plane* $\sigma \geqslant \sigma_0$ $(\sigma_0 < 1)$. *Then, as* $N \to \infty$,
$$\sum_{n=1}^{N} a_n \sim \frac{g(1)}{(k-1)!} N \log^{k-1} N.$$

This is merely a convenient statement of a well-known result. We have*
$$\sum_{n=1}^{N} a_n = \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} f(s) \frac{(N+\frac{1}{2})^s}{s}\, ds + O\left(\frac{N^\alpha}{T}\right) \quad (\alpha > 1;\ T > 0).$$
In the neighbourhood of $s = 1$, the integrand is
$$\left\{\frac{1}{(s-1)^k} + \ldots\right\}\{g(1)+\ldots\}(N+\tfrac{1}{2})\left\{1+(s-1)\log(N+\tfrac{1}{2})+\ldots+\right.$$
$$\left. + \frac{(s-1)^{k-1}}{(k-1)!} \log^{k-1}(N+\tfrac{1}{2}) + \ldots\right\}.$$
Hence the residue is
$$\frac{g(1)}{(k-1)!}(N+\tfrac{1}{2})\log^{k-1}(N+\tfrac{1}{2}) + O\{(N+\tfrac{1}{2})\log^{k-2}(N+\tfrac{1}{2})\}$$
$$= \frac{g(1)}{(k-1)!} N \log^{k-1} N + O(N \log^{k-2} N)$$
Accordingly, the term involving the integral is
$$\frac{1}{2\pi i}\left(\int_{\alpha-iT}^{\beta-iT} + \int_{\beta-iT}^{\beta+iT} + \int_{\beta+iT}^{\alpha+iT}\right) f(s) \frac{(N+\frac{1}{2})^s}{s}\, ds +$$
$$+ \frac{g(1)}{(k-1)!} N \log^{k-1} N + O(N \log^{k-2} N),$$

* See e.g. B. M. Wilson, loc. cit. 239.

where $\sigma_0 < \beta < 1$. Since

$$\zeta(s) = O(t^{\frac{1}{2}(1-\sigma)+\epsilon})$$

for $0 < \sigma < 1$, we have

$$\int_{\beta-iT}^{\beta+iT} f(s)\frac{(N+\frac{1}{2})^s}{s}\,ds = O(N^\beta T^{\frac{1}{2}k(1-\beta)+\epsilon}),$$

$$\int_{\beta\pm iT}^{\alpha\pm iT} f(s)\frac{(N+\frac{1}{2})^s}{s}\,ds = O(N^\alpha T^{\frac{1}{2}k(1-\beta)-1+\epsilon}).$$

Taking, say, $T = N^{1/k}$, and α and β sufficiently near to 1, the result follows.

Similarly, if $$f(s) = -\zeta^k(s)\zeta'(s)g(s),$$

we obtain $$\sum_{n=1}^{N} a_n \sim \frac{g(1)}{(k+1)!} N\log^{k+1}N,$$

and so with factors $\zeta''(s)$, etc.

Lemma β. *If* $$\sum_{n=1}^{N} a_n \sim CN\log^{k-1}N,$$

then $$\sum_{n=1}^{N} \frac{a_n}{n}\log^l n \sim C\frac{\log^{k+l}N}{k+l}.$$

We have

$$S_N = \sum_{n=1}^{N} a_n \log^l n = \sum_{n\leqslant N^{1-\epsilon}} + \sum_{N^{1-\epsilon}<n\leqslant N} \sim CN\log^{k+l-1}N.$$

Hence the given sum is

$$\sum_{n=1}^{N} \frac{S_n - S_{n-1}}{n} = \sum_{n=1}^{N} S_n\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{S_N}{N+1}$$

$$\sim \sum_{n=1}^{N} C\frac{\log^{k+l-1}n}{n}$$

$$\sim C\int_1^N \frac{\log^{k+l-1}x}{x}\,dx = C\frac{\log^{k+l}N}{k+l}.$$

3. By way of introduction we shall indicate briefly how the circle method would proceed in the cases of (1.2) and (1.3). We have

$$\frac{1}{2\pi}\int_0^{2\pi}|f_2(e^{i\theta-\delta})|^2e^{-ir\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sum_{m=1}^{\infty}d(m)e^{(i\theta-\delta)m}\sum_{n=1}^{\infty}d(n)e^{(-i\theta-\delta)n}e^{-ir\theta}\,d\theta$$

$$= \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}d(m)d(n)e^{-(m+n)\delta}\int_0^{2\pi}e^{i\theta(m-n-r)}\,d\theta$$

$$= \sum_{n=1}^{\infty}d(n)d(n+r)e^{-(2n+r)\delta}. \tag{3.1}$$

Let $1 \leqslant k \leqslant N$, $0 < h \leqslant k$, $(h,k) = 1$, so that h/k runs through the fractions of the Farey dissection of order N. Let

$$\left(\frac{h}{k}-\frac{\vartheta_1}{2\pi}, \frac{h}{k}+\frac{\vartheta_2}{2\pi}\right),$$

where $\vartheta_1 = \vartheta_1(h,k)$, $\vartheta_2 = \vartheta_2(h,k)$, be the Farey interval containing h/k. Then

$$\frac{1}{2\pi}\int_0^{2\pi}|f_2(e^{i\theta-\delta})|^2e^{-ir\theta}\,d\theta = \frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h}\int_{2\pi h/k-\vartheta_1}^{2\pi h/k+\vartheta_2}|f_2(e^{i\theta-\delta})|^2e^{-ir\theta}\,d\theta$$

$$= \frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h}e^{-2\pi irh/k}\int_{-\vartheta_1}^{\vartheta_2}|f_2(e^{2\pi ih/k+i\theta-\delta})|^2e^{-ir\theta}\,d\theta. \tag{3.2}$$

As $z \to 0$ through values with positive real parts,

$$f_2(e^{2\pi ih/k-z}) \sim \frac{\gamma-\log z-2\log k}{kz}. \tag{3.3}$$

An approximation to (3.2) should therefore be given by

$$\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h}e^{-2\pi irh/k}\int_{-\vartheta_1}^{\vartheta_2}\left|\frac{\gamma-\log(\delta-i\theta)-2\log k}{k(\delta-i\theta)}\right|^2e^{-ir\theta}\,d\theta$$

$$= \frac{1}{2\pi\delta}\sum_{k=1}^{N}\frac{1}{k^2}\sum_{h}e^{-2\pi irh/k}\int_{-\vartheta_1/\delta}^{\vartheta_2/\delta}\{(\gamma-\log\delta-2\log k)^2-$$

$$-(\gamma-\log\delta-2\log k)\log(1+\theta^2)+|\log(1-i\theta)|^2\}\frac{e^{-ir\delta\theta}}{1+\theta^2}\,d\theta. \tag{3.4}$$

Now $\vartheta_1 \geqslant \pi/kN$, $\vartheta_2 \geqslant \pi/kN$. Hence we can replace the limits by $(-\infty,\infty)$ with error

$$O\int_{\pi/\delta kN}^{\infty}\{(\gamma-\log\delta-2\log k)^2+\ldots\}\frac{d\theta}{1+\theta^2} = O\left(\delta kN\log^2\frac{1}{\delta}\right),$$

assuming that $N = O(\delta^{-A})$, $\delta kN < A$. Also

$$\int_{-\infty}^{\infty} \frac{e^{-ir\delta\theta}}{1+\theta^2}\,d\theta = \pi e^{-r\delta}, \qquad \int_{-\infty}^{\infty} \frac{\log(1+\theta^2)}{1+\theta^2} e^{-ir\delta\theta}\,d\theta = I(\delta),$$

$$\int_{-\infty}^{\infty} \frac{|\log(1-i\theta)|^2}{1+\theta^2} e^{-ir\delta\theta}\,d\theta = J(\delta),$$

where $I(\delta)$ and $J(\delta)$ are bounded as $\delta \to 0$. Hence (3.4) is

$$\frac{1}{2\pi\delta}\sum_{k=1}^{N}\frac{1}{k^2}\sum_{h} e^{-2\pi irh/k}\{\pi e^{-r\delta}(\gamma-\log\delta-2\log k)^2-$$

$$-I(\delta)(\gamma-\log\delta-2\log k)+J(\delta)\}+\frac{1}{2\pi\delta}\sum_{k=1}^{N}\frac{1}{k^2}\sum_{h} O\Bigl(\delta kN\log^2\frac{1}{\delta}\Bigr)$$

$$= \frac{1}{2\pi\delta}\sum_{k=1}^{N}\frac{c_k(r)}{k^2}\{\pi e^{-r\delta}(\gamma-\log\delta-2\log k)^2-$$

$$-I(\delta)(\gamma-\log\delta-2\log k)+J(\delta)\}+O\Bigl(N^2\log^2\frac{1}{\delta}\Bigr), \tag{3.5}$$

where

$$c_k(r) = \sum_{h} e^{-2\pi irh/k}$$

is Ramanujan's sum. For a fixed r, $c_k(r)$ is bounded. Hence, if we take, say,

$$N = \Bigl[\frac{1}{\delta^{\frac{1}{2}}}\log^{-1}\frac{1}{\delta}\Bigr],$$

the above expression is asymptotic to*

$$\frac{1}{2\delta}\log^2\frac{1}{\delta}\sum_{k=1}^{\infty}\frac{c_k(r)}{k^2} = \frac{3}{\pi^2}\frac{\sigma(r)}{r}\frac{1}{\delta}\log^2\frac{1}{\delta},$$

giving (1.3).

In the case $r = 0$, $c_k(r)$ is replaced by $\phi(k)$, the number of values of h. Hence (3.5) is replaced by

$$\frac{1}{2\pi\delta}\sum_{k=1}^{N}\frac{\phi(k)}{k^2}\{\pi(\gamma-\log\delta-2\log k)^2-I(\delta)(\gamma-\log\delta-2\log k)+J(\delta)\}+$$

$$+O\Bigl(N^2\log^2\frac{1}{\delta}\Bigr)$$

$$= \frac{1}{2\delta}\sum_{k=1}^{N}\frac{\phi(k)}{k^2}\Bigl(\log\frac{1}{\delta}-2\log k\Bigr)^2+O\Bigl(\frac{1}{\delta}\log^2\frac{1}{\delta}\Bigr)+O\Bigl(N^2\log^2\frac{1}{\delta}\Bigr).$$

* Hardy and Wright, *An Introduction to the Theory of Numbers* (Oxford, 1938), Th. 293.

Since
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^{s+1}} = \frac{\zeta(s)}{\zeta(s+1)},$$

Lemma α gives
$$\sum_{n=1}^{N} \frac{\phi(n)}{n} \sim \frac{N}{\zeta(2)} = \frac{6N}{\pi^2}.$$

From this and Lemma β we obtain

$$\sum_{k=1}^{N} \frac{\phi(k)}{k^2}\left(\log^2\frac{1}{\delta} - 4\log\frac{1}{\delta}\log k + 4\log^2 k\right)$$
$$\sim \frac{6}{\pi^2}\left(\log^2\frac{1}{\delta}\log N - 4\log\frac{1}{\delta}\,\frac{\log^2 N}{2} + 4\,\frac{\log^3 N}{3}\right).$$

Taking $N = [\delta^{-\frac{1}{2}}]$, $\log N \sim \frac{1}{2}\log 1/\delta$, this

$$\sim \frac{6}{\pi^2}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right)\log^3\frac{1}{\delta} = \frac{1}{\pi^2}\log^3\frac{1}{\delta},$$

and (1.2) follows.

The result is correct since, as in B. M. Wilson's paper,

$$\sum_{n=1}^{N} d^2(n) \sim \frac{N\log^3 N}{\pi^2},$$

e.g. by Lemma α with

$$f(s) = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

For a rigorous proof by the circle method we should have to prove that the error involved in using (3.3) in the above way is negligible. In view of Estermann's analysis there would be no great difficulty in doing this.

4. Before proceeding to the case of $f_3(z)$ we shall prove some more lemmas.

LEMMA γ. *Let*
$$b_n = \sum_{\nu|n} d\left(\frac{n}{\nu}\right)\frac{1}{\nu^a},$$

where a is a fixed positive number. Then

$$\sum_{n=1}^{N} b_n = O(N\log N), \tag{4.1}$$

$$\sum_{n=1}^{N} b_n^2 = O(N\log^3 N). \tag{4.2}$$

We have
$$\sum_{n=1}^{\infty}\frac{b_n}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\nu|n}d\left(\frac{n}{\nu}\right)\frac{1}{\nu^a}$$
$$=\sum_{m=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{1}{(m\nu)^s}\frac{d(m)}{\nu^a}$$
$$=\sum_{m=1}^{\infty}\frac{d(m)}{m^s}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{s+a}}$$
$$=\zeta^2(s)\zeta(s+a).$$
Then (4.1) follows from Lemma α. Also
$$\sum_{n=1}^{\infty}\frac{b_n}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-2}\left(1-\frac{1}{p^{s+a}}\right)^{-1}$$
$$=\prod_p\left(1+\frac{2}{p^s}+\frac{3}{p^{2s}}+\ldots\right)\left(1+\frac{1}{p^{s+a}}+\frac{1}{p^{2s+2a}}+\ldots\right)$$
$$=\prod_p\left(1+\frac{2+p^{-a}}{p^s}+\frac{3+2p^{-a}+p^{-2a}}{p^{2s}}+\ldots\right).$$
Hence
$$\sum_{n=1}^{\infty}\frac{b_n^2}{n^s}=\prod_p\left\{1+\frac{(2+p^{-a})^2}{p^s}+\frac{(3+2p^{-a}+p^{-2a})^2}{p^{2s}}+\ldots\right\}$$
$$=\prod_p\left\{1+\frac{4}{p^s}+O\left(\frac{1}{p^{\sigma+a}}\right)+O\left(\frac{1}{p^{2\sigma}}\right)\right\}$$
$$=\zeta^4(s)\prod_p\left\{1+O\left(\frac{1}{p^{\sigma+a}}\right)+O\left(\frac{1}{p^{2\sigma}}\right)\right\}.$$
The last factor is regular for $\sigma>1-a$, $\sigma>\frac{1}{2}$. Hence (4.2) follows from Lemma α.

LEMMA δ. *Let*
$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},\qquad g(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s},\qquad \ldots,$$
where $a_n=O(n^\epsilon)$, $b_n=O(n^\epsilon)$, ... *Let*
$$F(s)=\zeta^2(s)f(s+1)g(s+1)\ldots=\sum_{n=1}^{\infty}\frac{\alpha_n}{n^s}.$$

Then

$$\sum_{n=1}^{N} \alpha_n = O(N \log N),$$

$$\sum_{n=1}^{N} \alpha_n^2 = O(N \log^3 N).$$

If
$$f(s)g(s) = \sum_{n=1}^{\infty} \frac{k_n}{n^s},$$

then
$$k_n = \sum_{\nu|n} a_n b_{n/\nu} = O\left\{\sum_{\nu|n} \nu^\epsilon \left(\frac{n}{\nu}\right)^\epsilon\right\} = O\{n^\epsilon d(n)\} = O(n^\epsilon)$$

(and similarly if there are more factors). Hence

$$\alpha_n = \sum_{\nu|n} d\left(\frac{n}{\nu}\right)\frac{k_\nu}{\nu} = O\left\{\sum_{\nu|n} d\left(\frac{n}{\nu}\right)\nu^{\epsilon-1}\right\},$$

and the result follows from Lemma γ.

Corollary. If the factor $\zeta^2(s)$ is replaced by $\zeta(s)\zeta'(s)$, the results are of orders $O(N\log^2 N)$ and $O(N \log^5 N)$ respectively, since an extra factor $\log(n/\nu)$ occurs in the sum for α_n.

Lemma ϵ.
$$\sum_{n=1}^{\infty} \frac{\phi(nk)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} k \prod_{p|k} \frac{1-p^{-1}}{1-p^{-s}}.$$

Let* $k = \prod_p p^l$. Then

$$\sum_{n=1}^{\infty} \frac{\phi(nk)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{\phi(p^{l+m})}{p^{ms}}.$$

If $l > 0$,

$$\sum_{m=0}^{\infty} \frac{\phi(p^{l+m})}{p^{ms}} = \sum_{m=0}^{\infty} \frac{p^{l+m}(1-p^{-1})}{p^{ms}} = \frac{p^l(1-p^{-1})}{1-p^{1-s}},$$

and

$$\sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} = 1 + \sum_{m=1}^{\infty} \frac{p^m(1-p^{-1})}{p^{ms}} = 1 + \frac{1-p^{-1}}{p^{s-1}-1} = \frac{1-p^{-s}}{1-p^{1-s}}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{\phi(nk)}{n^s} = \prod_p \frac{1-p^{-s}}{1-p^{1-s}} \prod_{p|k} p^l \frac{1-p^{-1}}{1-p^{-s}} = \frac{\zeta(s-1)}{\zeta(s)} k \prod_{p|k} \frac{1-p^{-1}}{1-p^{-s}}.$$

* See M. M. Crum, 'On some Dirichlet series': *J. of London Math. Soc.* 15 (1940), 10–15 (§ 4).

LEMMA ζ. *For any* $f(n)$

$$\sum_{\substack{(h,k)=1\\ h\leqslant k}} f(h) = \sum_{\nu|k} \mu(\nu) \sum_{m\leqslant k/\nu} f(m\nu).$$

On the right-hand side, $m\nu = h$ occurs if $\nu|h$, $\nu|k$, i.e. $\nu|(h,k)$. Hence the sum is

$$\sum_{h\leqslant k} f(h) \sum_{\nu|(h,k)} \mu(\nu) = \sum_{\substack{h\leqslant k\\ (h,k)=1}} f(h).$$

LEMMA η. *Let*

$$\psi(n) = \sum_{\nu|n} \frac{\phi(\nu)}{\nu}, \qquad \psi_1(n) = \sum_{\nu|n} \frac{\phi(\nu)\log\nu}{\nu}, \qquad \psi_2(n) = \sum_{\nu|n} \frac{\phi(\nu)\log^2\nu}{\nu}.$$

Then

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(s+1)},$$

$$\sum_{n=1}^{\infty} \frac{\psi_1(n)}{n^s} = -\frac{\zeta(s)\zeta'(s)}{\zeta(s+1)} + \frac{\zeta^2(s)\zeta'(s+1)}{\zeta^2(s+1)},$$

and

$$\sum_{n=1}^{\infty} \frac{\psi_2(n)}{n^s}$$

$$= \frac{\zeta(s)\zeta''(s)}{\zeta(s+1)} - \frac{2\zeta(s)\zeta'(s)\zeta'(s+1)}{\zeta^2(s+1)} + \frac{2\zeta^2(s)\zeta'^2(s+1)}{\zeta^3(s+1)} - \frac{\zeta^2(s)\zeta''(s+1)}{\zeta^2(s+1)}.$$

For

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\nu|n} \frac{\phi(\nu)}{\nu} = \sum_{m=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{(m\nu)^s} \frac{\phi(\nu)}{\nu}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{\nu=1}^{\infty} \frac{\phi(\nu)}{\nu^{s+1}} = \zeta(s)\frac{\zeta(s)}{\zeta(s+1)},$$

and similarly

$$\sum_{n=1}^{\infty} \frac{\psi_1(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{\nu=1}^{\infty} \frac{\phi(\nu)\log\nu}{\nu^{s+1}} = \zeta(s)\left\{-\frac{d}{ds}\frac{\zeta(s)}{\zeta(s+1)}\right\},$$

$$\sum_{n=1}^{\infty} \frac{\psi_2(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{\nu=1}^{\infty} \frac{\phi(\nu)\log^2\nu}{\nu^{s+1}} = \zeta(s)\frac{d^2}{ds^2}\frac{\zeta(s)}{\zeta(s+1)}.$$

LEMMA θ. $\psi_1(n) = \frac{1}{2}\psi(n)\log n + \omega(n)$, *where*

$$\sum_{n=1}^{N} \omega^2(n) = O(N\log^3 N).$$

For
$$\sum_{n=1}^{\infty}\frac{\psi_1(n)}{n^s}=-\frac{1}{2}\frac{d}{ds}\frac{\zeta^2(s)}{\zeta(s+1)}+\frac{1}{2}\frac{\zeta^2(s)\zeta'(s+1)}{\zeta^2(s+1)}$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\psi(n)\log n}{n^s}+\sum_{n=1}^{\infty}\frac{\omega(n)}{n^s},$$
say, and the result follows from Lemma δ.

It might be supposed that, similarly, we could approximate to $\psi_2(n)$ by $A\psi(n)\log^2 n$ with some A, but subsequent lemmas show that this is not true.

5. The function f_3. The formula corresponding to (3.3) is*
$$f_3(e^{2\pi ih/k-z})\sim\frac{c_1\{\frac{1}{2}\log^2 z+\gamma\log z+\frac{1}{2}\Gamma''(1)\}-c_2(\log z+\gamma)+c_3}{z},\tag{5.1}$$
where c_1, c_2, c_3 depend on k only. Their values are given by
$$c_1=\frac{1}{k}\sum_{\kappa|k}\frac{\phi(\kappa)}{\kappa}=\frac{\psi(k)}{k},\tag{5.2}$$
$$c_2=\frac{1}{k}\sum_{\kappa|k}\frac{2(\gamma-\log\kappa)\phi(\kappa)}{\kappa}+\sum_{\kappa|k}\frac{1}{\kappa}\sum_{\alpha}\lambda_{\alpha k/\kappa,k},\tag{5.3}$$
and $c_3=c_{3,1}+c_{3,2}$, where
$$c_{3,1}=\sum_{\kappa|k}\sum_{\alpha}\left\{\frac{\mu_{\alpha k/\kappa,k}}{\kappa}+\frac{2(\gamma-\log\kappa)\lambda_{\alpha k/\kappa,k}}{\kappa}+\frac{\log^2\kappa-3\gamma\log\kappa+\gamma^2+2\gamma_1}{k\kappa}\right\},$$
$$c_{3,2}=-\frac{1}{k}\sum_{\kappa|k}\sum_{b=1}^{\kappa-1}\lambda_{b,\kappa}\frac{\Lambda(\kappa_1)\phi(\kappa)}{\phi(\kappa_1)}\quad\left(\kappa_1=\frac{\kappa}{(\kappa,b)}\right).$$
Here α runs through values less than κ and prime to κ,
$$\lambda_{k,k}=\frac{\gamma-\log k}{k},\qquad\mu_{k,k}=\frac{\gamma_1-\gamma\log k+\frac{1}{2}\log^2 k}{k},$$
and, if $a<k$,
$$\lambda_{a,k}-\lambda_{k,k}=\frac{1}{a}-\frac{1}{k}+\frac{1}{a+k}-\ldots,\qquad\mu_{a,k}-\mu_{k,k}=-\frac{\log a}{a}+\frac{\log k}{k}-\ldots.$$

* E. C. Titchmarsh, 'On a series of Lambert's type': *J. of London Math. Soc.* 13 (1938), 248–53.

We shall next transform these expressions into more convenient forms. We have

$$c_2 = -\frac{2}{k}\psi_1(k)+O\left\{\frac{d(k)}{k}\right\}+\sum_{\kappa|k}\frac{1}{\kappa}\sum_{\alpha}\left\{\frac{\kappa}{\alpha k}-\frac{\log k}{k}+O\left(\frac{1}{k}\right)\right\}$$

$$= -\frac{2}{k}\psi_1(k)+\frac{1}{k}\sum_{\kappa|k}\sum_{\alpha}\frac{1}{\alpha}-\frac{\log k\,\psi(k)}{k}+O\left\{\frac{d(k)}{k}\right\}.$$

Now, by Lemma ζ,

$$\sum_{\alpha}\frac{1}{\alpha}=\sum_{\nu|\kappa}\mu(\nu)\sum_{m\leqslant\kappa/\nu}\frac{1}{m\nu}=\sum_{\nu|\kappa}\frac{\mu(\nu)}{\nu}\left\{\log\frac{\kappa}{\nu}+O(1)\right\}$$

$$=\frac{\log\kappa\,\phi(\kappa)}{\kappa}+O\left(\sum_{\nu|\kappa}\frac{\log\nu+1}{\nu}\right).$$

Hence

$$c_2 = -\frac{\psi_1(k)}{k}-\frac{\log k\,\psi(k)}{k}+O\left(\frac{1}{k}\sum_{\kappa|k}\sum_{\nu|\kappa}\frac{\log\nu+1}{\nu}\right)+O\left\{\frac{d(k)}{k}\right\}$$

$$= c_2'+c_2'',$$

say, where

$$c_2' = -\frac{3\log k\,\psi(k)}{2k} \tag{5.4}$$

and c_2'' is the remainder. Since $(\log\nu)/\nu = O(\nu^{-\frac{1}{2}})$,

$$kc_2'' = O\{|\omega(k)|\}+O\left\{\sum_{\nu|k}d\left(\frac{k}{\nu}\right)\nu^{-\frac{1}{2}}\right\}+O\{d(k)\},$$

$$\sum_{k=1}^{N}(kc_2'')^2 = O(N\log^3 N) \tag{5.5}$$

by Lemmas γ and θ and (1.2). Also

$$c_{3,1} = \sum_{\kappa|k}\sum_{\alpha}\left\{\frac{1}{\kappa}\left(-\frac{\log\alpha k/\kappa}{\alpha k/\kappa}+\frac{\log^2 k}{2k}\right)+\frac{2\gamma}{\alpha k}-\right.$$

$$\left.-\frac{2\log\kappa}{\kappa}\left(\frac{1}{\alpha k/\kappa}-\frac{\log k}{k}\right)+\frac{\log^2\kappa}{k\kappa}+O\left(\frac{\log k}{k\kappa}\right)\right\}$$

$$= -\frac{1}{k}\sum_{\kappa|k}\sum_{\alpha}\frac{\log\alpha k\kappa}{\alpha}+\frac{\log^2 k\,\psi(k)}{2k}+\frac{2\log k\,\psi_1(k)}{k}+$$

$$+\frac{\psi_2(k)}{k}+O\left\{\frac{d(k)\log k}{k}\right\}.$$

By Lemma ζ,

$$\sum_{\alpha} \frac{\log \alpha}{\alpha} = \sum_{\nu|\kappa} \mu(\nu) \sum_{m \leqslant \kappa/\nu} \frac{\log m\nu}{m\nu} = \sum_{\nu|\kappa} \frac{\mu(\nu)}{\nu} \sum_{m \leqslant \kappa/\nu} \left(\frac{\log m}{m} + \frac{\log \nu}{m}\right)$$

$$= \sum_{\nu|\kappa} \frac{\mu(\nu)}{\nu} \left\{\frac{1}{2} \log^2 \frac{\kappa}{\nu} + \log \nu \log \frac{\kappa}{\nu} + O(\log \nu)\right\}$$

$$= \sum_{\nu|\kappa} \frac{\mu(\nu)}{\nu} \{\tfrac{1}{2} \log^2 \kappa + O(\log^2 \nu)\} = \frac{\log^2 \kappa\, \phi(\kappa)}{2\kappa} + O\left(\sum_{\nu|\kappa} \frac{\log^2 \nu + 1}{\nu}\right).$$

Hence

$$c_{3,1} = -\frac{1}{k}\left\{\tfrac{3}{2}\psi_2(k) + O\left(\sum_{\nu|k} d\left(\frac{k}{\nu}\right) \frac{\log^2 \nu + 1}{\nu}\right) + \log k\, \psi_1(k) + \right.$$

$$\left. + O\left(\log k \sum_{\nu|k} d\left(\frac{k}{\nu}\right) \frac{\log \nu + 1}{\nu}\right)\right\} + \frac{\log^2 k\, \psi(k)}{2k} + \frac{2 \log k\, \psi_1(k)}{k} +$$

$$+ \frac{\psi_2(k)}{k} + O\left\{\frac{d(k) \log k}{k}\right\} = c'_{3,1} + c''_{3,1},$$

where

$$c'_{3,1} = \frac{\log^2 k}{k} \psi(k) - \frac{1}{2k} \psi_2(k) \tag{5.6}$$

and

$$\sum_{k=1}^{N} (kc''_{3,1})^2 = O(N \log^5 N). \tag{5.7}$$

Again

$$c_{3,2} = -\frac{1}{k} \sum_{\kappa|k} \sum_{b=1}^{\kappa-1} \left\{\frac{1}{b} - \frac{\log \kappa}{\kappa} + O\left(\frac{1}{\kappa}\right)\right\} \frac{\Lambda(\kappa_1)\phi(\kappa)}{\phi(\kappa_1)}.$$

Let $(\kappa, b) = r$, $\kappa = \kappa_1 r$, $b = b_1 r$. Then, for a given κ_1, b runs through multiples of κ/κ_1 less than the κ_1th. Hence

$$c_{3,2} = -\frac{1}{k} \sum_{\kappa|k} \sum_{\kappa_1|\kappa} \frac{\Lambda(\kappa_1)\phi(\kappa)}{\phi(\kappa_1)} \sum_{(b_1,\kappa_1)=1,\, b_1<\kappa_1} \left\{\frac{\kappa_1}{b_1 \kappa} - \frac{\log \kappa}{\kappa} + O\left(\frac{1}{\kappa}\right)\right\}.$$

It is easily seen that, if $\kappa_1 = p^m$,

$$\sum_{(b_1,\kappa_1)=1,\, b_1<\kappa_1} \frac{1}{b_1} = \left(1 - \frac{1}{p}\right) \log \kappa_1 + O(1).$$

Hence

$$c_{3,2} = -\frac{1}{k} \sum_{\kappa|k} \sum_{p^m|\kappa} \frac{\log p\, \phi(\kappa)}{p^m(1-1/p)} \left\{\frac{p^m}{\kappa}\left(1 - \frac{1}{p}\right) \log \frac{p^m}{\kappa} + O\left(\frac{p^m}{\kappa}\right)\right\}$$

$$= -\frac{1}{k} \sum_{\kappa|k} \frac{\phi(\kappa)}{\kappa} \sum_{p^m|\kappa} \left\{\log p \log \frac{p^m}{\kappa} + O(\log p)\right\}.$$

The O-term contributes

$$O\left\{\frac{1}{k}\sum_{\kappa|k}\frac{\phi(\kappa)}{\kappa}\log\kappa\right\} = O\left\{\frac{\log k}{k}d(k)\right\}.$$

If $j(k)$ is what is obtained by omitting the O-term, then

$$\sum_{k=1}^{\infty}\frac{j(k)}{k^{s-1}} = -\sum_{k=1}^{\infty}\frac{1}{k^s}\sum_{\kappa|k}\frac{\phi(\kappa)}{\kappa}\sum_{p^m|\kappa}\log p\log\frac{p^m}{\kappa}$$

$$= -\zeta(s)\sum_{\kappa=1}^{\infty}\frac{\phi(\kappa)}{\kappa^{s+1}}\sum_{p^m|\kappa}\log p\log\frac{p^m}{\kappa}$$

$$= \zeta(s)\sum_{n=1}^{\infty}\sum_{p,m}\frac{\phi(np^m)}{(np^m)^{s+1}}\log p\log n.$$

Now

$$\sum_{n=1}^{\infty}\frac{\phi(np^m)\log n}{n^{s+1}} = -\frac{d}{ds}\sum_{n=1}^{\infty}\frac{\phi(np^m)}{n^{s+1}}$$

$$= -\frac{d}{ds}\frac{\zeta(s)}{\zeta(s+1)}p^m\frac{1-p^{-1}}{1-p^{-s}}$$

$$= -\frac{\zeta'(s)}{\zeta(s+1)}p^m+\ldots,$$

where the leading term contributes

$$-\frac{\zeta(s)\zeta'(s)}{\zeta(s+1)}\sum_{p,m}\frac{\log p}{p^{ms}} = \frac{\zeta'^2(s)}{\zeta(s+1)}.$$

Comparing this with the formula

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{\psi(k)\log^2 k}{k^s}-\sum_{k=1}^{\infty}\frac{\psi_2(k)}{k^s} = \frac{\zeta'^2(s)}{\zeta(s+1)}-\frac{\zeta^2(s)\zeta'^2(s+1)}{\zeta^3(s+1)}-\frac{3}{2}\frac{\zeta^2(s)\zeta''(s+1)}{\zeta^2(s+1)}$$

it is now easily shown that

$$kj(k) = \tfrac{1}{2}\psi(k)\log^2 k-\psi_2(k)+m(k),$$

where

$$\sum_{k=1}^{N}m^2(k) = O(N\log^5 N).$$

Altogether we obtain

$$c_{3,2} = c'_{3,2}+c''_{3,2},$$

where

$$c'_{3,2} = \frac{\psi(k)\log^2 k}{2k}-\frac{\psi_2(k)}{k} \tag{5.8}$$

and

$$\sum_{k=1}^{N}(kc''_{3,2})^2 = O(N\log^5 N). \tag{5.9}$$

6. Lemma ι. $$\sum_{n=1}^{N} \frac{\phi(n)\psi(n)}{n} \sim A_2 N \log N.$$

We have

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \prod_p \frac{1-p^{-s-1}}{(1-p^{-s})^2} = \prod_p \left(1+\frac{2-p^{-1}}{p^s}+\frac{3-2p^{-1}}{p^{2s}}+\ldots\right).$$

Hence

$$\sum_{n=1}^{\infty} \frac{\phi(n)\psi(n)}{n^{s+1}} = \prod_p \left\{1+\sum_{m=1}^{\infty} \frac{\phi(p^m)\psi(p^m)}{p^{m(s+1)}}\right\}$$

$$= \prod_p \left\{1+\sum_{m=1}^{\infty} \left(1-\frac{1}{p}\right)\left(m+1-\frac{m}{p}\right)\frac{1}{p^{ms}}\right\}$$

$$= \prod_p \left\{1+\left(1-\frac{1}{p}\right)\frac{1}{p^s}\left(2-\frac{1}{p}-\frac{1}{p^s}\right)\left(1-\frac{1}{p^s}\right)^{-2}\right\}$$

$$= \zeta^2(s) \prod_p \left(1-\frac{3}{p^{s+1}}+\frac{1}{p^{s+2}}+\frac{1}{p^{2s+1}}\right).$$

Thus by Lemma α

$$\sum_{n=1}^{N} \frac{\phi(n)\psi(n)}{n} \sim N \log N \prod_p \left(1-\frac{3}{p^2}+\frac{2}{p^3}\right) = A_2 N \log N.$$

Lemma κ. *Let* $$\psi(n,x) = \sum_{\nu|n} \frac{\phi(\nu)}{\nu^x}.$$

Then

$$\sum_{n=1}^{\infty} \frac{\phi(n)\psi(n,x)}{n^{s+1}} = \zeta(s)\zeta(s+x-1) \prod_p \left(1-\frac{1}{p^{s+1}}-\frac{2}{p^{s+x}}+\frac{1}{p^{s+x+1}}+\frac{1}{p^{2s+x}}\right).$$

The function $\psi(n,x)$ is multiplicative, and

$$\psi(p^m,x) = 1+\frac{p-1}{p^x}+\frac{p(p-1)}{p^{2x}}+\ldots+\frac{p^{m-1}(p-1)}{p^{mx}}$$

$$= 1+\frac{p-1}{p^x}\,\frac{1-p^{m(1-x)}}{1-p^{1-x}}$$

$$= \frac{1}{p^x-p}\left(p^x-1-\frac{p-1}{p^{m(x-1)}}\right).$$

Hence

$$\sum_{n=1}^{\infty}\frac{\phi(n)\psi(n,x)}{n^{s+1}}=\prod_p\left\{1+\sum_{m=1}^{\infty}\frac{\phi(p^m)\psi(p^m,x)}{p^{m(s+1)}}\right\}$$

$$=\prod_p\left\{1+\sum_{m=1}^{\infty}\frac{p-1}{p}\frac{1}{p^x-p}\left(p^x-1-\frac{p-1}{p^{m(x-1)}}\right)\frac{1}{p^{ms}}\right\}$$

$$=\prod_p\left\{1+\frac{p-1}{p(p^x-p)}\left(\frac{p^x-1}{p^s-1}-\frac{p-1}{p^{s+x-1}-1}\right)\right\}$$

$$=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{1}{p^{s+x-1}}\right)^{-1}\left(1-\frac{1}{p^{s+1}}-\frac{2}{p^{s+x}}+\frac{1}{p^{s+x+1}}+\frac{1}{p^{2s+x}}\right).$$

Lemma λ. $$\sum_{n=1}^{N}\frac{\phi(n)\psi_2(n)}{n}\sim\frac{A_2}{3}N\log^3N.$$

We have $$\psi_2(n)=\left[\frac{\partial^2}{\partial x^2}\psi(n,x)\right]_{x=1}=\psi_{xx}(n,1).$$

Writing the result of the last lemma as

$$F(s,x)=\zeta(s)\zeta(s+x-1)G(s,x),$$

we have

$$F_{xx}(s,x)=\zeta(s)\zeta''(s+x-1)G(s,x)+2\zeta(s)\zeta'(s+x-1)G_x(s,x)+$$
$$+\zeta(s)\zeta(s+x-1)G_{xx}(s,x).$$

Hence

$$\sum_{n=1}^{\infty}\frac{\phi(n)\psi_2(n)}{n^{s+1}}=F_{xx}(s,1)$$
$$=\zeta(s)\zeta''(s)G(s,1)+2\zeta(s)\zeta'(s)G_x(s,1)+\zeta^2(s)G_{xx}(s,1).$$

Since $G(s,1)$, $G_x(s,1)$, $G_{xx}(s,1)$ are regular and bounded for $\sigma\geqslant\sigma_0>0$, and $G(1,1)=A_2$, the result follows from Lemma α.

7. Sums involving $d(n)\,d_3(n)$. We have

$$d(p^m)=m+1,\qquad d_3(p^m)=\tfrac{1}{2}(m+1)(m+2).$$

Hence

$$\sum_{n=1}^{\infty}\frac{d(n)\,d_3(n)}{n^s}=\prod_p\sum_{m=0}^{\infty}\frac{d(p^m)d_3(p^m)}{p^{ms}}$$

$$=\prod_p\sum_{m=0}^{\infty}\frac{(m+1)^2(m+2)}{2p^{ms}}=\prod_p\frac{1+2p^{-s}}{(1-p^{-s})^4}$$

$$=\zeta^6(s)\prod_p\left(1-\frac{1}{p^s}\right)^2\left(1+\frac{2}{p^s}\right)=\zeta^6(s)\prod_p\left\{1+O\left(\frac{1}{p^{2\sigma}}\right)\right\}.$$

So by Lemma α
$$\sum_{n=1}^{N} d(n)\,d_3(n) \sim \frac{A_2}{5!} N\log^5 N,$$
and (1.7) follows by partial summation.

Applying the circle method to this problem, we have
$$\sum_{n=1}^{\infty} d(n)\,d_3(n)e^{-2n\delta} = \frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h}\int\limits_{2\pi h/k-\vartheta_1}^{2\pi h/k+\vartheta_2} f_2(e^{i\theta-\delta})f_3(e^{-i\theta-\delta})\,d\theta.$$
According to our hypothesis, we replace this by
$$\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h}\int\limits_{-\vartheta_1}^{\vartheta_2}\frac{\gamma-\log(\delta-i\theta)-2\log k}{k(\delta-i\theta)}\times$$
$$\times\frac{c_1\{\frac{1}{2}\log^2(\delta+i\theta)+\gamma\log(\delta+i\theta)+\frac{1}{2}\Gamma''(1)\}-c_2\log(\delta+i\theta)+c_3}{\delta+i\theta}\,d\theta.$$
Replacing the limits by $(-\infty,\infty)$, and rejecting obviously negligible terms in the numerator, we obtain
$$\frac{1}{2\pi}\sum_{k=1}^{N}\frac{\phi(k)}{k}\int\limits_{-\infty}^{\infty}\frac{(-\log\delta-2\log k)(\frac{1}{2}c_1\log^2\delta-c_2\log\delta+c_3)}{\delta^2+\theta^2}\,d\theta$$
$$=\frac{1}{2\delta}\sum_{k=1}^{N}\frac{\phi(k)}{k}\left(\log\frac{1}{\delta}-2\log k\right)\left(\tfrac{1}{2}c_1\log^2\frac{1}{\delta}+c_2\log\frac{1}{\delta}+c_3\right).$$
Replacing c_2 by c_2', and c_3 by $c_{3,1}'+c_{3,2}'$, we obtain
$$\frac{1}{2\delta}\sum_{k=1}^{N}\frac{\phi(k)}{k^2}\left(\log\frac{1}{\delta}-2\log k\right)\left\{\tfrac{1}{2}\psi(k)\log^2\frac{1}{\delta}-\frac{3}{2}\log k\,\psi(k)\log\frac{1}{\delta}+\right.$$
$$\left.+\tfrac{3}{2}\log^2 k\,\psi(k)-\tfrac{3}{2}\psi_2(k)\right\}$$
$$=\frac{1}{4\delta}\log^3\frac{1}{\delta}\sum_{k=1}^{N}\frac{\phi(k)\psi(k)}{k^2}-\frac{5}{4\delta}\log^2\frac{1}{\delta}\sum_{k=1}^{N}\frac{\phi(k)\psi(k)\log k}{k^2}+$$
$$+\frac{9}{4\delta}\log\frac{1}{\delta}\sum_{k=1}^{N}\frac{\phi(k)\psi(k)\log^2 k}{k^2}-\frac{3}{2\delta}\sum_{k=1}^{N}\frac{\phi(k)\psi(k)\log^3 k}{k^2}-$$
$$-\frac{3}{4\delta}\log\frac{1}{\delta}\sum_{k=1}^{N}\frac{\phi(k)\psi_2(k)}{k^2}+\frac{3}{2\delta}\sum_{k=1}^{N}\frac{\phi(k)\psi_2(k)\log k}{k^2}$$
$$\sim\frac{A_2}{\delta}\left(\frac{1}{4}\log^3\frac{1}{\delta}\,\frac{\log^2 N}{2}-\frac{5}{4}\log^2\frac{1}{\delta}\,\frac{\log^3 N}{3}+\frac{9}{4}\log\frac{1}{\delta}\,\frac{\log^4 N}{4}-\right.$$
$$\left.-\frac{3}{2}\,\frac{\log^5 N}{5}-\frac{3}{4}\log\frac{1}{\delta}\,\frac{\log^4 N}{12}+\frac{3}{2}\,\frac{\log^5 N}{15}\right).$$

If, as before, $\log N \sim \frac{1}{2}\log 1/\delta$, this is asymptotic to

$$\frac{A_2}{\delta}\log^5\frac{1}{\delta}\left(\frac{1}{8.4}-\frac{5}{12.8}+\frac{9}{16.16}-\frac{3}{10.32}-\frac{1}{16.16}+\frac{1}{10.32}\right)=\frac{A_2}{240}\frac{1}{\delta}\log^5\frac{1}{\delta}.$$

The circle method therefore gives the correct result in this case.

8. Sums involving $d_3(n)\,d(n+r)$. We have

$$\sum_{n=0}^{\infty} d_3(n)\,d(n+r)e^{-(2n+r)\delta}=\frac{1}{2\pi}\int\limits_0^{2\pi} f_2(e^{i\theta-\delta})f_3(e^{-i\theta-\delta})e^{-ir\theta}\,d\theta$$

$$=\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h}\int\limits_{2\pi h/k-\vartheta_1}^{2\pi h/k+\vartheta_2} f_2(e^{i\theta-\delta})f_3(e^{-i\theta-\delta})e^{-ir\theta}\,d\theta.$$

We approximate to this by

$$\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h} e^{-2\pi irh/k}\int\limits_{-\vartheta_1}^{\vartheta_2}\frac{\gamma-\log(\delta-i\theta)-2\log k}{k(\delta-i\theta)}\times$$

$$\frac{c_1\{\frac{1}{2}\log^2(\delta+i\theta)+\gamma\log(\delta+i\theta)+\frac{1}{2}\Gamma''(1)\}-c_2\{\log(\delta+i\theta)+\gamma\}+c_3}{\delta+i\theta}e^{-ir\theta}\,d\theta.$$

The highest term will be

$$\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h} e^{-2\pi irh/k}\int\limits_{-\infty}^{\infty}\frac{\log 1/\delta}{k(\delta-i\theta)}\,\frac{\frac{1}{2}c_1\log^2 1/\delta}{\delta+i\theta}e^{-ir\theta}\,d\theta$$

$$\sim\frac{1}{4\delta}\log^3\frac{1}{\delta}\sum_{k=1}^{N}c_k(r)\frac{c_1}{k}\sim\frac{1}{4\delta}\log^3\frac{1}{\delta}\sum_{k=1}^{\infty}\frac{c_k(r)\psi(k)}{k^2}.$$

The sum is equal to

$$\prod_p\left\{1+\sum_{m=1}^{\infty}\frac{c_{p^m}(r)\psi(p^m)}{p^{2m}}\right\}.$$

If $p\nmid r$, the factor involving p is

$$1-\frac{\psi(p)}{p^2}=1-\frac{2}{p^2}+\frac{1}{p^3}.$$

If $r=p^a\ldots$, the factor is

$$1+\frac{(p-1)(2-1/p)}{p^2}+\ldots+\frac{(p^a-p^{a-1})(a+1-a/p)}{p^{2a}}-\frac{p^a\{a+2-(a+1)/p\}}{p^{2a+2}}$$

$$=1+\frac{2}{p}-\frac{a+2}{p^{a+1}}-\frac{2}{p^{a+2}}+\frac{a+1}{p^{a+3}},$$

which is positive, since it is greater than

$$\left(1+\frac{2}{p}\right)\left(1-\frac{a+2}{p^{a+1}}\right).$$

Hence (1.10) follows formally.

9. LEMMA μ. $$\sum_{n=1}^{N}\frac{\phi(n)\psi^2(n)}{n}\sim\frac{A_1}{6}N\log^3 N.$$

We have

$$\sum_{n=1}^{\infty}\frac{\phi(n)\psi^2(n)}{n^{s+1}}=\prod_p\left\{1+\left(1-\frac{1}{p}\right)\left(2-\frac{1}{p}\right)^2\frac{1}{p^s}+\left(1-\frac{1}{p}\right)\left(3-\frac{2}{p}\right)^2\frac{1}{p^{2s}}+\dots\right\}$$

$$=\prod_p\left(1+\frac{4}{p^s}+\dots\right)=\zeta^4(s)g(s),$$

where $$g(s)=\prod_p\left\{1+O\left(\frac{1}{p^{2\sigma}}\right)+O\left(\frac{1}{p^{\sigma+1}}\right)\right\}.$$

Also

$$g(1)=\prod_p\left\{1+\left(1-\frac{1}{p}\right)\left(2-\frac{1}{p}\right)^2\frac{1}{p}+\left(1-\frac{1}{p}\right)\left(3-\frac{2}{p}\right)^2\frac{1}{p^2}+\dots\right\}\left(1-\frac{1}{p}\right)^4$$

$$=\prod_p\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p}\right)^4=A_1.$$

The result therefore follows from Lemma α.

LEMMA ν. *Let*

$$F(s,x,y)=\sum_{n=1}^{\infty}\frac{\phi(n)\psi(n,x)\psi(n,y)}{n^{s+1}}.$$

Then

$$F(s,x,y)=\zeta(s)\zeta(s+x-1)\zeta(s+y-1)\zeta(s+x+y-2)G(s,x,y),$$

where $$G(s,x,y)=\prod_p\left(1+\frac{Q}{p^{4s+2x+2y+3}}\right),$$

$Q=Q(p^x,p^y,p^s,p)$ *is a polynomial of degree* 9 *in its arguments, and*

$$G(1,1,1)=A_1.$$

We have

$$F(s,x,y)=\prod_p\left\{1+\frac{\phi(p)\psi(p,x)\psi(p,y)}{p^{s+1}}+\dots\right\},$$

and the bracket is

$$1+\sum_{m=1}^{\infty}\frac{p^{m-1}(p-1)}{p^{m(s+1)}}\frac{p^{x}-1-(p-1)p^{-m(x-1)}}{p^{x}-p}\frac{p^{y}-1-(p-1)p^{-m(y-1)}}{p^{y}-p}$$

$$=1+\frac{p-1}{p(p^{x}-p)(p^{y}-p)}\sum_{m=1}^{\infty}\left\{\frac{(p^{x}-1)(p^{y}-1)}{p^{ms}}-\frac{(p-1)(p^{y}-1)}{p^{m(s+x-1)}}-\right.$$

$$\left.-\frac{(p-1)(p^{x}-1)}{p^{m(s+y-1)}}+\frac{(p-1)^{2}}{p^{m(s+x+y-2)}}\right\}$$

$$=1+\frac{p-1}{p(p^{x}-p)(p^{y}-p)}\left\{\frac{(p^{x}-1)(p^{y}-1)}{p^{s}-1}-\frac{(p-1)(p^{y}-1)}{p^{s+x-1}-1}-\right.$$

$$\left.-\frac{(p-1)(p^{x}-1)}{p^{s+y-1}-1}+\frac{(p-1)^{2}}{p^{s+x+y-2}-1}\right\}.$$

Let us write temporarily $p^{1-x}=a$, $p^{1-y}=b$, $p^{s}=z$. Then this is

$$1+\frac{p-1}{p^{3}(1-a)(1-b)}\left\{\frac{(p-a)(p-b)}{z-1}-\frac{a^{2}(p-1)(p-b)}{z-a}-\right.$$

$$\left.-\frac{b^{2}(p-1)(p-a)}{z-b}+\frac{a^{2}b^{2}(p-1)^{2}}{z-ab}\right\}.$$

If we write $p-a=(p-1)+(1-a)$, etc., the bracket is

$$(p-1)^{2}\left(\frac{1}{z-1}-\frac{a^{2}}{z-a}-\frac{b^{2}}{z-b}+\frac{a^{2}b^{2}}{z-ab}\right)+$$

$$+(p-1)\left\{\frac{2-a-b}{z-1}-\frac{a^{2}(1-b)}{z-a}-\frac{b^{2}(1-a)}{z-b}\right\}+\frac{(1-a)(1-b)}{z-1}$$

$$=\frac{\begin{array}{r}(p-1)^{2}(1-a)(1-b)\{(1+a)(1+b)z^{3}-(a+b+3ab+ab^{2}+a^{2}b)z^{2}+\\ +ab(1+a)(1+b)z-a^{2}b^{2}\}\end{array}}{(z-1)(z-a)(z-b)(z-ab)}+$$

$$+\frac{(p-1)(1-a)(1-b)\{(2+a+b)z^{2}-2(a+b+ab)z+2ab\}}{(z-1)(z-a)(z-b)}+$$

$$+\frac{(1-a)(1-b)}{z-1}.$$

Hence the general factor in $F(s, x, y)$ is equal to

$$1+\frac{(p-1)^3\{(1+a)(1+b)z^3-(a+b+3ab+ab^2+a^2b)z^2+ +ab(1+a)(1+b)z-a^2b^2\}}{p^3(z-1)(z-a)(z-b)(z-ab)}+$$

$$+\frac{(p-1)^2\{(2+a+b)z^2-2(a+b+ab)z+2ab\}}{p^3(z-1)(z-a)(z-b)}+\frac{p-1}{p^3(z-1)}$$

$$=1+\frac{(1+a)(1+b)}{z}+\text{terms of degree } -2 \text{ in } p \text{ and } z \text{ together.}$$

Hence it is of the form

$$\frac{p^3z^4+\text{terms of degree 5}}{p^3(z-1)(z-a)(z-b)(z-ab)}$$

$$=\frac{p^{4s+3}+\ldots}{p^3(p^s-1)(p^s-p^{1-x})(p^s-p^{1-y})(p^s-p^{2-x-y})}$$

$$=\frac{1+\text{terms of degree } -2}{(1-p^{-s})(1-p^{1-s-x})(1-p^{1-s-y})(1-p^{2-s-x-y})}.$$

The first part now follows. Also, if in the general factor we take the denominator as $p^3(z-1)(z-a)(z-b)(z-ab)$, and then put $a=b=1$, $z=p$ in the numerator, we obtain

$$p^3(p-1)^4+(p-1)^3(4p^3-7p^2+4p-1)+(p-1)^3(4p^2-6p+2)+(p-1)^4$$
$$=(p-1)^4(p^3+4p^2+p).$$

Hence $$G(1,1,1)=\prod_p\frac{(p-1)^4(p^3+4p^2+p)}{p^7}=A_1.$$

LEMMA ξ. $$\sum_{n=1}^{N}\frac{\phi(n)\psi(n)\psi_2(n)}{n}\sim\frac{A_1}{20}N\log^5 N.$$

We have

$$\sum_{n=1}^{\infty}\frac{\phi(n)\psi(n)\psi_2(n)}{n^{s+1}}=F_{xx}(s,1,1)=\left[\frac{\partial^2}{\partial x^2}\zeta^2(s)\zeta^2(s+x-1)G(s,x,1)\right]_{x=1}$$
$$=2\{\zeta^2(s)\zeta'^2(s)+\zeta^3(s)\zeta''(s)\}G(s,1,1)+$$
$$+4\zeta^3(s)\zeta'(s)G_x(s,1,1)+\zeta^4(s)G_{xx}(s,1,1).$$

In the neighbourhood of $s=1$,

$$\zeta^2(s)\zeta'^2(s)+\zeta^3(s)\zeta''(s)=\frac{3}{(s-1)^6}+\ldots.$$

Hence by Lemma α (extension) the term involving $G(s,1,1)$ contributes

$$\frac{6G(1,1,1)}{5!}N\log^5 N = \frac{A_1}{20}N\log^5 N.$$

Similarly the other terms contribute $O(N\log^4 N)$ and $O(N\log^3 N)$.

LEMMA o. $$\sum_{n=1}^{N}\frac{\phi(n)\psi_2^2(n)}{n} \sim \frac{19A_1}{1260}N\log^7 N.$$

We have $$\sum_{n=1}^{\infty}\frac{\phi(n)\psi_2^2(n)}{n^{s+1}} = F_{xxyy}(s,1,1).$$

As in the previous cases, the main term will be the contribution of

$$\begin{aligned}
\zeta(s)G(s,x,y)&\frac{\partial^4}{\partial x^2\partial y^2}\zeta(s+x-1)\zeta(s+y-1)\zeta(s+x+y-2)\\
&= \zeta(s)G(s,x,y)\{\zeta''(s+x-1)\zeta''(s+y-1)\zeta(s+x+y-2)+\\
&\quad +\zeta''(s+x-1)\zeta(s+y-1)\zeta''(s+x+y-2)+\\
&\quad +\zeta(s+x-1)\zeta''(s+y-1)\zeta''(s+x+y-2)+\\
&\quad +2\zeta''(s+x-1)\zeta'(s+y-1)\zeta'(s+x+y-2)+\\
&\quad +2\zeta'(s+x-1)\zeta''(s+y-1)\zeta'(s+x+y-2)+\\
&\quad +4\zeta'(s+x-1)\zeta'(s+y-1)\zeta''(s+x+y-2)+\\
&\quad +2\zeta'(s+x-1)\zeta(s+y-1)\zeta'''(s+x+y-2)+\\
&\quad +2\zeta(s+x-1)\zeta'(s+y-1)\zeta'''(s+x+y-2)+\\
&\quad +\zeta(s+x-1)\zeta(s+y-1)\zeta''''(s+x+y-2)\}.
\end{aligned}$$

Putting $x = y = 1$, this gives

$$\begin{aligned}
G(s,1,1)&\{3\zeta(s)\zeta''^2(s)+8\zeta(s)\zeta'^2(s)\zeta''(s)+4\zeta^2(s)\zeta'(s)\zeta'''(s)+\zeta^3(s)\zeta''''(s)\}\\
&= \frac{76A_1}{(s-1)^8}+\dots.
\end{aligned}$$

The leading term in the sum is therefore

$$\frac{76A_1 N\log^7 N}{7!} = \frac{19A_1}{1260}N\log^7 N.$$

10. Sums involving $d_3^2(n)$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{d_3^2(n)}{n^s} &= \prod_p \sum_{m=0}^{\infty}\{\tfrac{1}{2}(m+1)(m+2)\}^2\frac{1}{p^{ms}}\\
&= \prod\left(1+\frac{9}{p^s}+\dots\right) = \zeta^9(s)g(s),
\end{aligned}$$

where $g(s)$ is regular and bounded for $\sigma \geqslant \sigma_0 > \frac{1}{2}$. Also

$$g(1) = \prod_p \left[\left(1-\frac{1}{p}\right)^9 \sum_{m=0}^{\infty} \{\tfrac{1}{2}(m+1)(m+2)\}^2 \frac{1}{p^m}\right]$$

$$= \prod_p \left(1-\frac{1}{p}\right)^4 \left(1+\frac{4}{p}+\frac{1}{p^2}\right) = A_1.$$

Hence, by Lemma α,

$$\sum_{n=1}^{N} d_3^2(n) \sim \frac{A_1}{8!} N \log^8 N,$$

and (1.5) follows by partial summation.

On the other hand

$$\sum_{n=1}^{\infty} d_3^2(n) e^{-2n\delta} = \frac{1}{2\pi} \sum_{k=1}^{N} \sum_h \int_{-\vartheta_1}^{\vartheta_2} |f_3(e^{2\pi i h/k + i\theta - \delta})|^2 \, d\theta.$$

If we could use the approximation (5.1) for f_3, this would be replaced by

$$\frac{1}{2\pi} \sum_{k=1}^{N} \sum_h \int_{-\vartheta_1}^{\vartheta_2} \frac{\begin{array}{r}|c_1\{\frac{1}{2}\log^2(\delta-i\theta)+\gamma\log(\delta-i\theta)+\frac{1}{2}\Gamma''(1)\}- \\ -c_2\{\log(\delta-i\theta)+\gamma\}+c_3|^2\end{array}}{\delta^2+\theta^2} \, d\theta.$$

As in the previous cases, we replace the limits by $(-\infty, \infty)$ and reject all but the highest powers of $\log \delta$. This gives

$$\frac{1}{2\delta} \sum_{k=1}^{N} \phi(k) \left(\frac{1}{2} c_1 \log^2 \frac{1}{\delta} + c_2 \log \frac{1}{\delta} + c_3\right)^2.$$

Approximating to c_2 and c_3 as before, we obtain

$$\frac{1}{2\delta} \sum_{k=1}^{N} \phi(k) \left\{\frac{1}{2} \frac{\psi(k)}{k} \log^2 \frac{1}{\delta} - \frac{3}{2} \frac{\log k}{k} \psi(k) \log \frac{1}{\delta} + \frac{3}{2} \frac{\log^2 k}{k} \psi(k) - \frac{3}{2k} \psi_2(k)\right\}^2$$

$$= \frac{1}{8\delta} \sum_{k=1}^{N} \frac{\phi(k)\psi^2(k)}{k^2} \left(\log^4 \frac{1}{\delta} - 6 \log k \log^3 \frac{1}{\delta} + 15 \log^2 k \log^2 \frac{1}{\delta} - \right.$$

$$\left. - 18 \log^3 k \log \frac{1}{\delta} + 9 \log^4 k\right) -$$

$$- \frac{3}{4\delta} \sum_{k=1}^{N} \frac{\phi(k)\psi(k)\psi_2(k)}{k^2} \left(\log^2 \frac{1}{\delta} - 3 \log k \log \frac{1}{\delta} + 3 \log^2 k\right) +$$

$$+ \frac{9}{8\delta} \sum_{k=1}^{N} \frac{\phi(k)\psi_2^2(k)}{k^2}.$$

By Lemmas α, β, μ, ξ, and o, this is asymptotic to

$$\frac{A_1}{48\delta}\left(\frac{\log^4 N}{4}\log^4\frac{1}{\delta}-6\frac{\log^5 N}{5}\log^3\frac{1}{\delta}+15\frac{\log^6 N}{6}\log^2\frac{1}{\delta}-\right.$$

$$\left.-18\frac{\log^7 N}{7}\log\frac{1}{\delta}+9\frac{\log^8 N}{8}\right)-$$

$$-\frac{3A_1}{80\delta}\left(\frac{\log^6 N}{6}\log^2\frac{1}{\delta}-3\frac{\log^7 N}{7}\log\frac{1}{\delta}+3\frac{\log^8 N}{8}\right)+\frac{9}{8\delta}\frac{19A_1}{1260}\frac{\log^8 N}{8}.$$

If $\log N \sim \frac{1}{2}\log 1/\delta$, this is asymptotic to

$$\frac{A_1}{\delta}\log^8\frac{1}{\delta}\left\{\frac{1}{48}\left(\frac{1}{4.2^4}-\frac{6}{5.2^5}+\frac{5}{2.2^6}-\frac{18}{7.2^7}+\frac{9}{8.2^8}\right)\right.$$

$$\left.-\frac{3}{80}\left(\frac{1}{6.2^6}-\frac{3}{7.2^7}+\frac{3}{8.2^8}\right)+\frac{19}{64.140.2^8}\right\}=\frac{17A_1}{2^{16}.21}\frac{1}{\delta}\log^8\frac{1}{\delta}.$$

11. Sums involving $d_3(n)\,d_3(n+r)$. We have

$$\sum_{n=0}^{\infty} d_3(n)d_3(n+r)e^{-(2n+r)\delta}$$

$$=\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h} e^{-2\pi irh/k}\int_{-\vartheta_1}^{\vartheta_2}|f_3(e^{2\pi ih/k+i\theta-\delta})|^2 e^{-ir\theta}\,d\theta.$$

If we could use (5.1), this would be replaced by

$$\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h} e^{-2\pi irh/k}\int_{-\vartheta_1}^{\vartheta_2}\frac{\begin{array}{r}|c_1\{\frac{1}{2}\log^2(\delta-i\theta)+\gamma\log(\delta-i\theta)+\frac{1}{2}\Gamma''(1)\}-\\ -c_2\{\log(\delta-i\theta)+\gamma\}+c_3|^2\end{array}}{\delta^2+\theta^2}e^{-ir\theta}\,d\theta$$

$$\sim\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{h} e^{-2\pi irh/k}\int_{-\infty}^{\infty}\frac{\frac{1}{4}c_1^2\log^4 1/\delta}{\delta^2+\theta^2}e^{-ir\theta}\,d\theta\sim\frac{1}{8\delta}\log^4\frac{1}{\delta}\sum_{k=1}^{\infty}\frac{c_k(r)\psi^2(k)}{k^2}.$$

This gives (1.9). The last series can be expressed as a product in the same way as that in § 8.

SIMPLE FOURIER TRANSFORMATIONS

By A. P. GUINAND (*R.C.A.F.*)

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1. Introduction

A function $K(x)$ is said to be a 'Fourier kernel' if, for all $f(x)$ satisfying appropriate conditions,*

$$g(x) = \int_0^\infty K(xt)f(t)\,dt \tag{1.1}$$

implies that

$$f(x) = \int_0^\infty K(xt)g(t)\,dt. \tag{1.2}$$

It can be shown that these results should be formally true if

$$K(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{K}(s)x^{-s}\,ds, \tag{1.3}$$

where

$$\mathfrak{K}(s)\mathfrak{K}(1-s) = 1. \tag{1.4}$$

A number of examples of transformations of the form (1.1) and (1.2) have been given, most of them involving circular, Bessel, or hypergeometric functions. In two previous papers I have shown that transformations involving simpler functions are of importance in the theories of summation formulae and prime numbers.†

In the present paper further examples of such simple transformations are given, and also some examples of an extension of the results to functions of several variables. Only formal derivations of the transformations are given, but conditions for the validity of each example are stated. The validity under these conditions may readily be verified in each case by direct substitution.‡

2. Transformations of functions of a single variable

It was previously noted that, if we put§

$$\mathfrak{K}(s) = \frac{s-\alpha}{s+\alpha-1},$$

* E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), 212–13.

† A. P. Guinand, *Quart. J. of Math.* (Oxford), 10 (1939), 38–44, and ibid. 13 (1942), 30–9, referred to as (A) and (B) respectively.

‡ Cf. (A), 43.

§ (B), (2.2) and (2.3).

we are led to the transformation

$$g(x) = \frac{1}{x}f\left(\frac{1}{x}\right)+(2\alpha-1)x^{\alpha-1}\int\limits_{1/x}^{\infty} t^{\alpha-1}f(t)\,dt \tag{2.1}$$

and its inverse if $R(\alpha) < \frac{1}{2}$, or

$$g(x) = \frac{1}{x}f\left(\frac{1}{x}\right)+(1-2\alpha)x^{\alpha-1}\int\limits_{0}^{1/x} t^{\alpha-1}f(t)\,dt \tag{2.2}$$

if $R(\alpha) > \frac{1}{2}$.

We obtain an extension of these results if we put

$$\mathfrak{K}(s) = P(s)/Q(s),$$

where $P(s)$ and $Q(s)$ are polynomials in s having no factor in common, and $\mathfrak{K}(s)$ satisfies (1.4). Suppose that the roots of $P(s)$ are $\delta_1, \delta_2,..., \delta_m$, and that the roots of $Q(s)$ are $\alpha_1, \alpha_2,..., \alpha_n$. We then have

$$\mathfrak{K}(s) = k\left\{\prod_{r=1}^{m}(s-\delta_r)\right\}\Big/\left\{\prod_{r=1}^{n}(s-\alpha_r)\right\}$$

for some constant k. Hence, by (1.4),

$$k^2\frac{\prod\limits_{r=1}^{m}(s-\delta_r)\prod\limits_{r=1}^{m}(1-s-\delta_r)}{\prod\limits_{r=1}^{n}(s-\alpha_r)\prod\limits_{r=1}^{n}(1-s-\alpha_r)} = 1$$

identically. Consequently, the factors $(s-\delta_r)$, $(1-s-\alpha_r)$ and $(1-s-\delta_r)$, $(s-\alpha_r)$ must cancel in pairs, and the degree m of $P(s)$ must be equal to the degree n of $Q(s)$, and $k = \pm 1$. Hence

$$\mathfrak{K}(s) = \pm\prod_{r=1}^{n}\left(\frac{1-s-\alpha_r}{s-\alpha_r}\right). \tag{2.3}$$

If we expand this expression in partial fractions, we shall have

$$\mathfrak{K}(s) = \pm 1+\sum_{r=1}^{n}\frac{A_r}{(s-\alpha_r)^{\beta_r+1}},$$

where β_r are non-negative integers and the A_r are constants. Let us take the positive sign for convenience, and omit the suffixes r. Then

$$\mathfrak{K}(s) = 1+\sum\frac{A}{(s-\alpha)^{\beta+1}}. \tag{2.4}$$

Now the term 1 leads to a term* $x^{-1}f(x^{-1})$ in the transform of $f(x)$, and

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{x^{-s}}{(s-\alpha)^{\beta+1}}\,ds$$

may be evaluated by moving the line of integration to the left or the right.

For $R(\alpha) < \frac{1}{2}$ this integral becomes

$$\frac{x^{-\alpha}(-\log x)^{\beta}}{\beta!} \quad (x < 1),$$

$$0 \qquad (x > 1),$$

and for $R(\alpha) > \frac{1}{2}$ it becomes

$$0 \qquad (x < 1),$$

$$\frac{-x^{-\alpha}(-\log x)^{\beta}}{\beta!} \quad (x > 1).$$

If $R(\alpha) = \frac{1}{2}$, the integral does not converge, so we will assume that such terms do not occur.

It follows that we are led to a transformation

$$g(x) = \frac{1}{x}f\left(\frac{1}{x}\right) + \sum_{R(\alpha)<\frac{1}{2}}\frac{Ax^{-\alpha}}{\beta!}\int_0^{1/x} t^{-\alpha}(-\log xt)^{\beta}f(t)\,dt -$$
$$- \sum_{R(\alpha)>\frac{1}{2}}\frac{Ax^{-\alpha}}{\beta!}\int_{1/x}^{\infty} t^{-\alpha}(-\log xt)^{\beta}f(t)\,dt. \quad (2.5)$$

We can verify the validity of the inverse formula by direct substitution, and we find:

THEOREM 1. *If the numbers* A, α, β *satisfy* (2.3) *and* (2.4), $x^{-\frac{1}{2}}f(x)$ *belongs to* $L(0,\infty)$, *and* $R(\alpha) \neq \frac{1}{2}$ *for all* α, *then* $g(x)$, *defined by* (2.5), *exists, and*

$$f(x) = \frac{1}{x}g\left(\frac{1}{x}\right) + \sum_{R(\alpha)<\frac{1}{2}}\frac{Ax^{-\alpha}}{\beta!}\int_0^{1/x} t^{-\alpha}(-\log xt)^{\beta}g(t)\,dt -$$
$$- \sum_{R(\alpha)>\frac{1}{2}}\frac{Ax^{-\alpha}}{\beta!}\int_{1/x}^{\infty} t^{-\alpha}(-\log xt)^{\beta}g(t)\,dt.$$

The conditions on $f(x)$ are necessary to justify the proof by direct substitution.

* E. C. Titchmarsh, loc. cit. 218.

Example. If we put

$$\mathfrak{K}(s) = \frac{\Gamma(s)\Gamma(1-s+a)}{\Gamma(s+a)\Gamma(1-s)},$$

where a is a positive integer, then all $\beta = 0$, and Theorem 1 gives

$$g(x) = \frac{1}{x}f\left(\frac{1}{x}\right) + \sum_{r=0}^{a-1} \frac{(-)^{r+a}(r+a)!}{(r!)^2(a-r-1)!} x^r \int_0^{1/x} t^r f(t)\, dt,$$

and the inverse formula holds if $x^{-\frac{1}{2}}f(x)$ belongs to $L(0,\infty)$.

3. Transformations of functions of several variables

The formal result quoted in the introduction may be extended to transforms of functions of two or more variables. For instance, if we have a function $\mathfrak{K}(r,s)$ satisfying

$$\mathfrak{K}(r,s)\mathfrak{K}(1-r,1-s) = 1 \tag{3.1}$$

and
$$K(x,y) = \frac{1}{(2\pi i)^2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{K}(r,s)x^{-r}y^{-s}\, drds,$$

then the inversion formula

$$g(x,y) = \int_0^\infty \int_0^\infty K(xu,yv)f(u,v)\, dudv,$$

$$f(x,y) = \int_0^\infty \int_0^\infty K(xu,yv)g(u,v)\, dudv$$

is formally true.

Obviously any function

$$\mathfrak{K}(r,s) = \mathfrak{K}_1(r)\mathfrak{K}_2(s),$$

where $\mathfrak{K}_1$ and $\mathfrak{K}_2$ satisfy (1.4), will satisfy (3.1) and give rise to a 'double Fourier kernel'

$$K(x,y) = K_1(x)K_2(y),$$

where K_1 and K_2 are Fourier kernels for functions of a single variable. However, we can construct examples of double or multiple Fourier kernels which cannot be split into factors in the above way. For example, if

$$\mathfrak{K}(r,s) = \frac{r+s-2}{r+s},$$

then (3.1) is satisfied, and we are led to the transformation

$$g(x,y) = \frac{1}{xy} f\left(\frac{1}{x},\frac{1}{y}\right) - \frac{2}{xy} \int_0^1 t f\left(\frac{t}{x},\frac{t}{y}\right) dt.$$

Similarly $$\mathfrak{K}(r,s) = \frac{r+s}{r+s-2}$$

gives us $$g(x,y) = \frac{1}{xy} f\left(\frac{1}{x},\frac{1}{y}\right) - \frac{2}{xy} \int_1^\infty f\left(\frac{t}{x},\frac{t}{y}\right) \frac{dt}{t}.$$

These transformations may be extended to functions of n variables, and a parameter α may be introduced as in (2.1) and (2.2). We obtain the following theorems.

THEOREM 2. *If* (i) $R(\alpha) < \frac{1}{2}$, (ii) $f(x_1, x_2,..., x_n)$ *is a function of n variables,* (iii) *the integrals*

$$\int_0^1 |f(a_1 t, a_2 t,..., a_n t)| t^{n-n\alpha-1}\, dt, \qquad \int_1^\infty |f(a_1 t, a_2 t,..., a_n t)| t^{n\alpha-1}\, dt$$

are absolutely convergent for all positive $a_1, a_2,..., a_n$, *and*

(iv) $$g(x_1, x_2,..., x_n) = \frac{1}{x_1 x_2 ... x_n} f\left(\frac{1}{x_1},\frac{1}{x_2},...,\frac{1}{x_n}\right) - \frac{n(1-2\alpha)}{x_1 x_2 ... x_n} \int_1^\infty f\left(\frac{t}{x_1},\frac{t}{x_2},...,\frac{t}{x_n}\right) t^{n\alpha-1}\, dt,$$

then the corresponding inverse formula holds for $f(x_1, x_2,..., x_n)$.

THEOREM 3. *If* (i) $R(\alpha) > \frac{1}{2}$, (ii) $f(x_1, x_2,..., x_n)$ *is a function of n variables,* (iii) *the integrals*

$$\int_0^1 |f(a_1 t, a_2 t,..., a_n t)| t^{n\alpha-1}\, dt, \qquad \int_1^\infty |f(a_1 t, a_2 t,..., a_n t)| t^{n-n\alpha-1}\, dt$$

are absolutely convergent for all positive $a_1, a_2,..., a_n$, *and*

(iv) $$g(x_1, x_2,..., x_n) = \frac{1}{x_1 x_2 ... x_n} f\left(\frac{1}{x_1},\frac{1}{x_2},...,\frac{1}{x_n}\right) - \frac{n(2\alpha-1)}{x_1 x_2 ... x_n} \int_0^1 f\left(\frac{t}{x_1},\frac{t}{x_2},...,\frac{t}{x_n}\right) t^{n\alpha-1}\, dt,$$

then the corresponding inverse formula holds for $f(x_1, x_2,..., x_n)$.

Another type of multiple transformation may be found corresponding to the transformations of functions of a single variable discussed in (A) in connexion with finite summation formulae.

For example, if we put

$$\mathfrak{K}(r,s) = \frac{a^{r-1}-b^{s-1}}{a^{-r}-b^{-s}} \quad (0 < a < b),$$

we obtain

THEOREM 4. *If* $0 < a < b$,

$$\sum_{n=1}^{\infty} |f(a^{-n}x, b^n y)| < K$$

for all positive x *and* y, *and*

$$g(x,y) = \frac{b-a}{xy} \sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n f\left(\frac{a^{n+1}}{x}, \frac{b^{-n+1}}{y}\right) - \frac{a}{xy} f\left(\frac{a}{x}, \frac{b}{y}\right),$$

then the corresponding inverse formula holds for $f(x,y)$. *Further, the 'finite summation formula'*

$$f(a,1)-f(1,b) = g(a,1)-g(1,b)$$

holds.

Again, these results may be verified by direct substitution.

In general, we can extend the above result to obtain inversion formulae for functions of n variables. The formulae would be of the form

$$g(x_1, x_2,..., x_n) = \sum_{r=0}^{\infty} \frac{c_r}{x_1 x_2 ... x_n} f\left(\frac{\alpha_r}{x_1}, \frac{\beta_r}{x_2}, ..., \frac{\omega_r}{x_n}\right),$$

together with the corresponding inverse formula with appropriate coefficients c_r, α_r, β_r,..., ω_r.

EXPANSIONS OF HYPERGEOMETRIC FUNCTIONS

By T. W. CHAUNDY (*Oxford*)

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1. In two recent articles (**2**), (**3**) in this *Journal* J. L. Burchnall and I gave a number of expansions of hypergeometric functions in series of hypergeometric functions. Their primary purpose was to throw light on the double hypergeometric functions by expressing them in terms of the elementary hypergeometric function. From the point of view of 'expansion for expansion's sake' one can supplement these by some other series which I collect here. I do not regard them as of great importance, but, having found them helpful in work in the hypergeometric field, I put them on record as possibly of use to others.

In the series of which I have spoken above* the parameters increase, in the successive terms, by increments of 1 or 2. More precisely, the series of simple hypergeometric functions fall into two types according as the parameters in the rth term are $(a+r, b+r, c+r)$ or $(a+r, b+r, c+2r)$; I shall temporarily distinguish these as the ***first*** and *second types* respectively.

I begin with two expansions (one of each type) of an elementary hypergeometric function in series of elementary hypergeometric functions, both sets of parameters being arbitrary:†

$$F(A,B;C;x)$$
$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r}{r!\,(c)_r}\, {}_4F_3\begin{bmatrix} A,\ B,\ c,\ -r \\ a,\ b,\ C \end{bmatrix} x^r F(a+r, b+r; c+r; x), \tag{1}$$

$$F(A,B;C;x) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r}{r!\,(c+r-1)_r} \times$$
$$\times\, {}_4F_3\begin{bmatrix} A,\ B,\ c+r-1,\ -r \\ a,\ b,\ C \end{bmatrix} x^r F(a+r, b+r; c+2r; x). \tag{2}$$

The formal proof is simple and suggests that the results are not very profound. On the right of (1) we can pick out the coefficient of $(A)_n(B)_n(c)_n/n!\,(a)_n(b)_n(C)_n$ as

$$\sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(-n)_r}{r!\,(c)_r} x^r F(a+r, b+r; c+r; x).$$

* More particularly (**2**) 253–7, (26)–(55).

† In the 'coefficient' ${}_4F_3$ the omission of the variable x indicates by a usual convention that $x = 1$.

Granted absolute convergence we can rearrange this into

$$\sum_{R=0}^{\infty} \frac{(a)_R(b)_R}{(c)_R} x^R \sum_{r=n}^{R} \frac{(-1)^{r-n}}{(r-n)!\,(R-r)!}.$$

By Vandermonde's theorem the inner sum vanishes unless $R = n$, and then it is unity. Thus the repeated sum reduces to $(a)_n(b)_n x^n/(c)_n$, and so the right-hand side of (1) is simply $F(A, B; C; x)$. This proves (1).

The proof of (2) similarly depends on the following lemma.

LEMMA 1.

$$\sum_{r=0}^{R} \frac{(-1)^r}{r!\,(R-r)!\,(c+r-1)_r(c+2r)_{R-r}} = \begin{cases} 0 & (R > 0), \\ 1 & (R = 0). \end{cases}$$

Proof. If $R = 0$, the series evidently reduces to its first term unity.

Now $\qquad (R-r)(c+r-1)+r(c+r+R-1) = R(c+2r-1),$

and so

$$\frac{R}{r!\,(R-r)!\,(c+r-1)_r(c+2r)_{R-r}} = \frac{(R-r)(c+r-1)+r(c+r+R-1)}{r!\,(R-r)!\,(c+r-1)_{R+1}}.$$

Thus, if $R \neq 0$, the series can be written as

$$\frac{1}{R}\left\{\sum_{r=0}^{R-1} \frac{(-1)^r}{r!\,(R-r-1)!\,(c+r)_R} - \sum_{r=1}^{R} \frac{(-1)^{r-1}}{(r-1)!\,(R-r)!\,(c+r-1)_R}\right\},$$

where the limits of summation have been adjusted to exclude zero terms. These two series evidently cancel, and the proof of the lemma is complete.

Then on the right of (2) we pick out the coefficient of

$$(A)_n(B)_n/n!\,(a)_n(b)_n(C)_n$$

which, granted absolute convergence, we rearrange as

$$\sum_{R=0}^{\infty} (a)_R(b)_R x^R \sum_{r=n}^{R} \frac{(-1)^{r-n}}{(r-n)!\,(R-r)!\,(c+r+n-1)_{r-n}(c+2r)_{R-r}}.$$

By the lemma, the inner sum vanishes unless $R = n$, and then it is unity. The repeated sum thus reduces simply to $(a)_n(b)_n x^n$, and (2) is proved.

2. In (2), as always in expansions of the 'second type', the parameter-difference '$a+b-c$' is constant as we proceed along the expansion. Thus we can use the elementary identity

$$(1-x)^{a+b-c}F(a, b; c; x) = F(c-a, c-b; c; x) \tag{3}$$

to obtain from (2) an expansion for

$$(1-x)^{a+b-c}F(A,B;C;x)$$

that differs from (2) itself only in the replacing of the hypergeometric function by

$$F(c-a+r,\ c-b+r;\ c+2r;\ x).$$

If now the two sets of parameters obey the condition

$$A+B-C = a+b-c, \tag{4}$$

the expansion represents just

$$F(C-A, C-B;\ C;\ x).$$

Changing $a,b,\ A,B$ to $c-a,\ c-b,\ C-A,\ C-B$ respectively we get a variant of (2) in the form

$$F(A,B;C;x) = \sum_{r=0}^{\infty} (-)^r \frac{(c-a)_r(c-b)_r}{r!\,(c+r-1)_r} \times$$

$$\times\, {}_4F_3\begin{bmatrix} C-A, & C-B, & c+r-1, & -r \\ c-a, & c-b, & C & \end{bmatrix} x^r F(a+r,b+r;c+2r;x).$$

Hence necessarily (by comparing coefficients)

$$(c-a)_r(c-b)_r\, {}_4F_3\begin{bmatrix} C-A, & C-B, & c+r-1, & -r \\ c-a, & c-b, & C & \end{bmatrix}$$

$$= (a)_r(b)_r\, {}_4F_3\begin{bmatrix} A, & B, & c+r-1, & -r \\ a, & b, & C & \end{bmatrix}.$$

This is otherwise evident, for under the condition (4) the ${}_4F_3$ are 'Saalschützian', and the above identity is one of of a set given by Whipple* for Saalschützian ${}_4F_3$.

In particular, if we write

$$A,\ B,\ C \equiv a,\ b+h,\ c+h,$$

the Saalschützian condition (4) is satisfied, and, moreover, the ${}_4F_3$ reduces to the Saalschützian ${}_3F_2$

$${}_3F_2\begin{bmatrix} b+h, & c+r-1, & -r \\ & b, & c+h \end{bmatrix}.$$

By Saalschütz's own theorem† this factorizes as

$$\frac{(c-b)_r(h-r+1)_r}{(c+h)_r(-b-r+1)_r}, \quad \text{i.e. } \frac{(c-b)_r(-h)_r}{(c+h)_r(b)_r}.$$

* Whipple (5), 537 (10.1); quoted by Bailey (1), 56, § 7.2 (1).

† (4), 279 (1); quoted by Bailey (1), 9, § 2.2 (1).

Then (2) gives the less general but simpler formula

$$F(a,b+h;c+h;x)$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r(c-b)_r(-h)_r}{r!\,(c+r-1)_r(c+h)_r}(-x)^r F(a+r,b+r;c+2r;x). \quad (5)$$

We may note the corollary given by $h=-b$:

$$1=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{r!\,(c+r-1)_r}(-x)^r F(a+r,b+r;c+2r;x), \quad (6)$$

which may be regarded as an inversion of the elementary hypergeometric series itself.

There is another simple expansion with restricted parameters, namely

$$F(a,b+h;c;x)=\sum_{r=0}^{\infty}\frac{(a)_r(h)_r}{r!\,(c)_r}x^r F(a+r,b;c+r;x). \quad (7)$$

It is evidently not covered by (1) or (2) since the parameters on the right proceed by different increments. The formal proof is immediate since, on the right, the coefficient of x^n is

$$\frac{(a)_n}{(c)_n}\sum_{r=0}^{n}\frac{(h)_r(b)_{n-r}}{r!\,(n-r)!},$$

and this is $(a)_n(b+h)_n/n!\,(c)_n$ by Vandermonde's theorem.

3. Double hypergeometric functions

Formulae analogous to (1) and (2) can easily be written down for Appell's double hypergeometric functions. Those of the 'first type' are

$$F^{(1)}[A;B,B';C;x,y]$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-)^{r+s}\frac{(a)_{r+s}(b)_r(b')_s}{r!\,s!\,(c)_{r+s}}F\begin{bmatrix}A,\ c:\ B,\ -r;\ B',\ -s\\ a,\ C:\quad b;\quad b'\end{bmatrix}\times$$
$$\times x^r y^s F^{(1)}[a+r+s;b+r,b'+s;c+r+s;x,y], \quad (8)$$

$$F^{(2)}[A;B,B';C,C';x,y]$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-)^{r+s}\frac{(a)_{r+s}(b)_r(b')_s}{r!\,s!\,(c)_r(c')_s}F\begin{bmatrix}A:\ B,\ c,\ -r;\ B',\ c',\ -s\\ a:\quad b,\ C;\quad b',\ C'\end{bmatrix}\times$$
$$\times x^r y^s F^{(2)}[a+r+s;b+r,b'+s;c+r,c'+s;x,y], \quad (9)$$

$$F^{(3)}[A,A';B,B';C;x,y]$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-)^{r+s}\frac{(a)_r(a')_s(b)_r(b')_s}{r!\,s!(c)_{r+s}}F\begin{bmatrix}c: A, & B, & -r; & A', & B', & -s\\ C: & a, & b; & & a', & b'\end{bmatrix}\times$$

$$\times x^r y^s F^{(3)}[a+r,a'+s;b+r,b'+s;c+r+s;x,y], \quad (10)$$

$$F^{(4)}[A,B;C,C';x,y]$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-)^{r+s}\frac{(a)_{r+s}(b)_{r+s}}{r!\,s!\,(c)_r(c')_s}F\begin{bmatrix}A, & B: & c, & -r; & c', & -s)\\ a, & b: & C; & & C' & \end{bmatrix}\times$$

$$\times x^r y^s F^{(4)}[a+r+s,b+r+s;c+r,c'+s;x,y]. \quad (11)$$

In (8), $F\begin{bmatrix}A, & c: & B, & -r; & B', & -s\\ a, & C: & b; & & b' & \end{bmatrix}$ denotes the double hypergeometric series*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(A)_{m+n}(c)_{m+n}(B)_m(-r)_m(B')_n(-s)_n}{m!\,n!\,(a)_{m+n}(C)_{m+n}(b)_m(b')_n},$$

the absent variables being both unity. So, in the other 'coefficient' F's, the colons mark off the 'double' parameters (i.e. those occurring with suffix $m+n$), the semicolons the 'simple' parameters (occurring with suffix m or n).† The pattern of the four formulae is clear: simple and double parameters keep that character throughout the formula. The formal proof in each case follows exactly the proof of (1), Vandermonde's theorem being used in the form

$$\sum_{r=m}^{R}\sum_{s=n}^{S}\frac{(-1)^{r+s-m-n}}{(r-m)!\,(R-r)!\,(s-n)!\,(S-s)!}=\begin{cases}1 & (R=m;\ S=n),\\ 0 & (R>m \ \text{ or } \ S>n).\end{cases}$$

Of the expansions of the 'second type' I give only that for $F^{(1)}$:

$$F^{(1)}[A;B,B';C;x,y]=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-)^{r+s}\frac{(a)_{r+s}(b)_r(b')_s}{r!\,s!\,(c+r+s-1)_{r+s}}\times$$

$$\times F\begin{bmatrix}A, & c+r+s-1: & B, & -r; & B', & -s\\ a, & C: & b; & & b' & \end{bmatrix}\times$$

$$\times x^r y^s F^{(1)}[a+r+s;b+r,b'+s;c+2r+2s;x,y]. \quad (12)$$

This differs from (8) above only in that on the right c is replaced by $c+2r+2s$ in $F^{(1)}$ and by $c+r+s-1$ in its coefficient. The expansion for $F^{(3)}$ is written down similarly from (10). The expansions for

* Terminating when, as now, r and s are positive integers.

† This rather conflicts with the accepted notation for $F^{(3)}$, which I find misleading.

$F^{(2)}$, $F^{(4)}$, where c, c' are simple parameters, are obtainable likewise from (9), (11) if we replace c by $c+2r$, $c+r-1$, and c' by $c'+2s$, $c'+s-1$.

In the proof of (12) and the corresponding expansion for $F^{(3)}$ we need the lemma, analogous to Lemma 1:

LEMMA 2.

$$\sum_{r=0}^{R}\sum_{s=0}^{S}\frac{(-1)^{r+s}}{r!\,(R-r)!\,s!\,(S-s)!\,(c+r+s-1)_{r+s}(c+2r+2s)_{R+S-r-s}}$$

vanishes unless R, S are both zero; and then it reduces to unity.

In the proof of the other two expansions the corresponding double sum in the lemma is expressible as the product of two simple sums and Lemma 1 suffices.

4. I conclude with a group of single expansions in $F^{(1)}$ and $F^{(3)}$ which have some relation to the identity (3) above. They are

$$(1-x)^{a+b-c}(1-y)^{a+b'-c}F^{(1)}[a;b,b';c;x,y]$$
$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r(c-b-b')_r}{r!\,(c)_{2r}}\times$$
$$\times x^r y^r F^{(1)}[c-a+r;c-b+r,c-b'+r;c+2r;x,y], \quad (13)$$

$$(1-x)^{a+b-c}F^{(1)}[a;b,b';c;x,y]$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r(c-a)_r(b')_r}{r!\,(c)_{2r}}x^r y^r F^{(3)}[c-a+r,a+r;c-b+r,b'+r;c+2r;x,y], \quad (14)$$

$$(1-x)^{a+b-c}F^{(3)}[a,c-a;b,b';c;x,y]$$
$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r(b')_r}{r!\,(c)_{2r}}x^r y^r F^{(1)}[c-a+r;c-b+r,b'+r;c+2r;x,y], \quad (15)$$

$$(1-x)^{a+b-c}F^{(3)}[a,a';b,b';c;x,y]=\sum_{r=0}^{\infty}\frac{(c-a-b)_r(a')_r(b')_r}{r!\,(c)_{2r}}\times$$
$$\times x^r y^r F^{(3)}[c-a+r,a'+r;c-b+r,b'+r;c+2r;x,y]. \quad (16)$$

To complete the group I add

$$(1-x)^{a+b-c}(1-y)^{a+b'-c}F^{(1)}[a;b,b';c;x,y]$$
$$=\sum_{r=0}^{\infty}\frac{(c-a)_r(c-b)_r(c-b')_r}{r!\,(c)_{2r}}\,{}_3F_2\left[\begin{matrix}a,\ c-b-b',\ -r\\ c-b,\ c-b'\end{matrix}\right]\times$$
$$\times x^r y^r F^{(3)}[c-a+r,c-a+r;c-b+r,c-b'+r;c+2r;x,y], \quad (17)$$

$$(1-x)^{a+b-c}(1-y)^{a+b'-c}F^{(3)}[a,a;b,b';c;x,y]$$
$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r(c-b-b')_r}{r!\,(c)_{2r}}\,{}_3F_2\left[\begin{matrix}b,\ b',\ -r\\ c-a,\ 1+b+b'-c-r\end{matrix}\right]\times$$
$$\times x^ry^rF^{(1)}[c-a+r;c-b+r,c-b'+r;c+2r;x,y], \quad (18)$$

$$(1-x)^{a+b-c}(1-y)^{a'+b'-c}F^{(3)}[a,a';b,b';c;x,y]$$
$$=\sum_{r=0}^{\infty}\frac{(c-a)_r(c-b)_r(c-a'-b')_r}{r!\,(c)_{2r}}\,{}_4F_3\left[\begin{matrix}c-a-b,\ a',\ b',\ -r\\ c-a,\ c-b,\ a'+b'-c-r+1\end{matrix}\right]\times$$
$$\times x^ry^rF^{(3)}[c-a+r,c-a'+r;c-b+r,c-b'+r;c+2r;x,y]. \quad (19)$$

In these last three series the coefficients are no longer simple gamma products but include also terminating hypergeometric functions of argument unity, as indeed do the coefficients in the general expansions (1), (2), and (8)–(11). There is certainly a case for extending the notion of hypergeometric series to take in series whose coefficients are thus generalized; I hope to consider such extended series in subsequent work. We may note that in (19) the coefficient on the right, which lacks apparent symmetry, can be written, in virtue of a theorem of Whipple's,* symmetrically but less concisely as

$$\frac{(c-a)_r(c-b)_r(c-a')_r(c-b')_r}{r!\,(c)_r(c)_{2r}}\,{}_7F_6\left[\begin{matrix}c,\ \tfrac{1}{2}c+\tfrac{1}{2},\ a,\ b,\ a',\ b',\ -r\\ \tfrac{1}{2}c-\tfrac{1}{2},\ c-a,\ c-b,\ c-a',\ c-b',\ c+r\end{matrix}\right]. \quad (20)$$

If in (13) we write $c = b+b'$, we get a closer analogy to (3) in the form

$$(1-x)^{a-b'}(1-y)^{a-b}F^{(1)}[a;b,b';b+b';x,y]$$
$$= F^{(1)}[b+b'-a;b',b;b+b';x,y], \quad (21)$$

but this is actually a variant of (3) itself, since we can write

$$F^{(1)}[a;b,b';b+b';x,y] = (1-y)^{-a}F\left(a,b;b+b';\frac{x-y}{1-y}\right).$$

5. One can make use of various rearrangements to show the formal equivalence of the two sides of the identities (13)–(19). I begin with the proof of (13). By (2) (30) we have

$$F^{(1)}[a;b,b';c;x,y]=\sum_{r=0}^{\infty}\frac{(a)_r(c-a)_r(b)_r(b')_r}{r!\,(c+r-1)_r(c)_{2r}}x^ry^rF(a+r,b+r;c+2r;x)\times$$
$$\times F(a+r,b'+r;c+2r;y).$$

* Whipple (6), 253 (7.7); quoted by Bailey (1), 29 (5).

and so, by (3) above,

$$(1-x)^{a+b-c}(1-y)^{a+b'-c}F^{(1)}[a;b,b';c;x,y]$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c-a)_r(b)_r(b')_r}{r!\,(c+r-1)_r(c)_{2r}}x^ry^rF(c-a+r,c-b+r;c+2r;x)\times$$
$$\times F(c-a+r,c-b'+r;c+2r;y)$$

$$={}_7F_6\left[\begin{matrix}c-1, & \frac{1}{2}c+\frac{1}{2}, & a, & b, & b', & -\delta, & -\delta' \\ & \frac{1}{2}c-\frac{1}{2}, & c-a, & c-b, & c-b', & c+\delta, & c+\delta'\end{matrix}\right]\times$$
$$\times F(c-a,c-b;c;x)F(c-a,c-b';c;y),$$

with the technique of (2) §§1, 2. Here the ${}_7F_6$ is 'well-poised' and, by Whipple's theorem just quoted, we can write it

$${}_4F_3\left[\begin{matrix}a, & c-b-b', & -\delta, & -\delta' \\ c-b, & c-b', & a-c+1-\delta-\delta'\end{matrix}\right]\nabla(c-a)\Delta(c),$$

where, as in (5),

$$\nabla(c-a)\equiv\frac{\Gamma(\delta+\delta'+c-a)\Gamma(c-a)}{\Gamma(\delta+c-a)\Gamma(\delta'+c-a)},\qquad \Delta(c)\equiv\frac{\Gamma(\delta+c)\Gamma(\delta'+c)}{\Gamma(\delta+\delta'+c)\Gamma(c)},$$

so that

$$\nabla(c-a)\Delta(c)F(c-a,c-b;c;x)F(c-a,c-b';c;y)$$
$$=F^{(1)}[c-a;c-b,c-b';c;x,y].$$

Thus now

$$(1-x)^{a+b-c}(1-y)^{a+b'-c}F^{(1)}[a;b,b';c;x,y]$$

$$={}_4F_3\left[\begin{matrix}a, & c-b-b', & -\delta, & -\delta' \\ c-b, & c-b', & a-c+1-\delta-\delta'\end{matrix}\right]F^{(1)}[c-a;c-b,c-b';c;x,y]$$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r(c-b-b')_r}{r!\,(c)_{2r}}x^ry^r\times$$
$$\times F^{(1)}[c-a+r;c-b+r,c-b'+r;c+2r;x,y],$$

by (2) (19).

Similarly, to prove (16) we use (2) 28 to expand its left-hand member as

$${}_6F_5\left[\begin{matrix}c-1, & \frac{1}{2}c+\frac{1}{2}, & a, & b, & -\delta, & -\delta'; \\ & \frac{1}{2}c-\frac{1}{2}, & c-a, & c-b, & c+\delta, & c+\delta';\end{matrix}\ -1\right]\times$$
$$\times F(c-a,c-b;c;x)F(a',b';c;y).$$

The ${}_6F_5$ is well-poised and reduces* to

$${}_3F_2\left[\begin{matrix}c-a-b, & -\delta, & -\delta' \\ c-a, & c-b\end{matrix}\right]\frac{\Gamma(c+\delta)\Gamma(c+\delta')}{\Gamma(c)\Gamma(c+\delta+\delta')}.$$

* Whipple (6), 252 (6.2); quoted by Bailey (1), 28 (2).

Thus

$$(1-x)^{a+b-c}F^{(3)}[a,a';b,b';c;x,y]$$
$$= {}_3F_2\left[\begin{matrix} c-a-b, \ -\delta, \ -\delta' \\ c-a, \ c-b \end{matrix}\right] F^{(3)}[c-a,a';c-b,b';c;x,y]$$
$$= \sum_{r=0}^{\infty} \frac{(c-a-b)_r(a')_r(b')_r}{r!\,(c)_{2r}} x^r y^r \times$$
$$\times F^{(3)}[c-a+r,a'+r;c-b+r,b'+r;c+2r;x,y],$$

which is (16).

To deduce (14), (15) we use (**2**) (32),

i.e. $F^{(1)}[a;b,b';c;x,y]$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r}{r!\,(c)_{2r}} x^r y^r F^{(3)}[a+r,a+r;b+r,b'+r;\,c+2r;\,x,y], \tag{22}$$

to replace $F^{(1)}$ by $F^{(3)}$ on one side or other of these identities, getting a double series in $F^{(3)}$. This, summed diagonally, reduces, after an application of Vandermonde's theorem, to the required simple series. We deduce (17), (18) similarly from (13), using in (18) the series inverse to (22), i.e. (**2**) (33); diagonal summation of the double series now gives a ${}_3F_2$ in place of the ${}_2F_1$ of Vandermonde's theorem. Finally, (19) comes from (16) and the similar series for $(1-y)^{a'+b'-c}F^{(3)}$.

6. Convergence conditions

6.1. The absolute convergence of the less general expansions (5)–(7) and (13)–(16) follows at once from the arguments used in (**2**) §§7–9. As there, it is sufficient to suppose parameters and arguments positive. By (**2**) (77),

$$F(a+r,b+r;c+2r;x) < \frac{(c)_{2r}}{(a)_r(b)_r} F(a,b;c;x)$$

and so (5), (6) above converge absolutely when $|x| < 1$. Similarly,*

$$F(a+r,b;c+r;x) < \frac{(c)_r}{(A)_r} F(a,b;c;x),$$

where $A \equiv \min(a,c)$. Thus (7) too converges absolutely when $|x| < 1$. Again, by (**2**) (81), (85),

$$F^{(1)}[a+r;b+r,b'+r;c+2r;x,y]$$
$$< \frac{(c)_{2r}}{(a)_r(A)_r}(1-xy)^{-r}F^{(1)}[a;b,b';c;x,y],$$

* By deduction from (**2**) 264 (i).

where $A \equiv \min(a,b,b')$,

$$F^{(3)}[a+r,b+r;a'+r,b'+r;c+2r;x,y] < \frac{(c)_{2r}}{(B)_r(B')_r} F^{(3)}[a,b;a',b';c;x,y],$$

where $B \equiv \min(b',c-b)$, $B' \equiv \min(b,c-b')$. Thus (13), (15) converge absolutely when $|xy| < \frac{1}{2}$; (14), (16) when $|xy| < 1$.

6.2. In the remaining series we have to take account, in the coefficients, of a terminating hypergeometric series one or more of whose parameters involve the parameter r of summation and we need some measure of the order of this factor as r diverges to infinity. It will be found always to have the order of a 'gamma product' $(\alpha)_r/(\beta)_r$ and it therefore introduces no new disturbing feature in the convergence. I begin for simplicity with

$${}_3F_2\begin{bmatrix} a_1, a_2, -r \\ c_1, 1-c_2-r \end{bmatrix}$$

which occurs (in different symbols) in (18). We can write this as

$$\frac{1}{(c_2)_r} \sum_{n=0}^{r} \binom{r}{n} \frac{(a_1)_n(a_2)_n(c_2)_{r-n}}{(c_1)_n}. \tag{23}$$

If a_1, a_2, c_1, c_2 are all positive, then, for sufficiently large n,

$$(a_1+n-1)(a_2+n-1) < (A+n-1)(c_1+n-1) \tag{24}$$

if $A > a_1+a_2-c_1$; and, in fact, we can choose A large enough to secure (24) when $n \geqslant 1$. Hence

$$(a_1)_n(a_2)_n/(c_1)_n < (A)_n. \tag{25}$$

Thus
$$\begin{aligned} {}_3F_2\begin{bmatrix} a_1, a_2, -r \\ c_1, 1-c_2-r \end{bmatrix} &< \frac{1}{(c_2)_r} \sum_{n=0}^{r} \binom{r}{n} (A)_n(c_2)_{r-n} \\ &= (A+c_2)_r/(c_2)_r, \end{aligned}$$

by Vandermonde's theorem. Even if a_1, a_2, c_1 are not all positive, the factors in (24) are positive for sufficiently large n, and it is enough to cut out a finite number of terms at the start of the series; similarly, if $c_2 < 0$, we need to cut out a finite number of terms at the end of the series. These excluded terms have all an order $(\alpha)_r/(\beta)_r$, so that (23) is always of this order. We can extend the inequality (24) to cover the factor

$${}_4F_3\begin{bmatrix} a_1, a_2, a_3, -r \\ c_1, c_2, 1-c_3-r \end{bmatrix}$$

represented in (19).

6.3. In ${}_3F_2\left[\begin{matrix}a_1, a_2, -r\\ c_1, c_2\end{matrix}\right]$, represented in (17), the terms of the finite series alternate in sign and the argument is less simple. I introduce the recurrence and difference operators E_1, E_2, ∇_1, ∇_2 defined by

$$(E_1)^r\frac{\Gamma(a_1)}{\Gamma(c_1)} \equiv \frac{\Gamma(a_1+r)}{\Gamma(c_1+r)}, \qquad \nabla_1 \equiv 1-E_1, \quad \text{etc.}$$

Dropping suffixes we notice that

$$\frac{\Gamma(c)}{\Gamma(a)}\nabla^r\frac{\Gamma(a)}{\Gamma(c)} = {}_2F_1(a, -r; c) = \frac{(c-a)_r}{(c)_r},$$

by Vandermonde's theorem, or by repeated operation with ∇.

Similarly,

$${}_3F_2\left[\begin{matrix}a_1, a_2, -r\\ c_1, c_2\end{matrix}\right] = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)}(1-E_1E_2)^r\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c_1)\Gamma(c_2)}.$$

Now $(1-E_1E_2)^r = (\nabla_1+E_1\nabla_2)^r = \sum\limits_{m+n=r}\binom{r}{m}\nabla_1^m\nabla_2^n E_1^n.$

Thus

$$(1-E_1E_2)^r\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c_1)\Gamma(c_2)} = \sum_{m+n=r}\binom{r}{m}\nabla_1^m\nabla_2^n\frac{\Gamma(a_1+n)\Gamma(a_2)}{\Gamma(c_1+n)\Gamma(c_2)}$$

$$= \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c_1)\Gamma(c_2)}\sum_{m+n=r}\binom{r}{m}\frac{(a_1)_n(c_1-a_1)_n(c_2-a_2)_n}{(c_1)_n(c_1+n)_m(c_2)_n},$$

and so*

$${}_3F_2\left[\begin{matrix}a_1, a_2, -r\\ c_1, c_2\end{matrix}\right] = \frac{1}{(c_1)_r}\sum_{m+n=r}\binom{r}{m}\frac{(c_1-a_1)_m(a_1)_n(c_2-a_2)_n}{(c_2)_n}. \tag{26}$$

This has the form of (23) and therefore the order of a gamma product. The argument can be extended to the ${}_4F_3$ in (1). We have $1-E_1E_2E_3 = \nabla_1+E_1\nabla_2+E_1E_2\nabla_3$, and this leads to

$${}_4F_3\left[\begin{matrix}a_1, a_2, a_3, -r\\ c_1, c_2, c_3\end{matrix}\right]$$

$$= \frac{1}{(c_1)_r}\sum_{m+n+p=r}\sum\binom{r}{m, n}\frac{(c_1-a_1)_m(c_2-a_2)_n(c_3-a_3)_p(a_1)_{n+p}(a_2)_p}{(c_2)_{n+p}(c_3)_p},$$

the first symbol in the summand denoting a multinomial coefficient. A double application of (23) then shows that the double sum has the

* The identity (26) is well known as an identity between two ${}_3F_2$, but I need here some *ad hoc* argument that can be extended to series of higher order.

order of a gamma product. This analysis is unsuitable for series of the type occurring in (2), for the sum corresponding to (26) then has alternating signs, the factor analogous to $(c_2-a_2)_n$ becoming $(c_2-a_2-r)_n$.

6.4. To discuss the order of the ${}_4F_3$ in (2) I broach a method that I hope to exploit more fully in subsequent work. Write

$$F_r \equiv \frac{(a)_r}{r!}\,{}_4F_3\left[\begin{matrix} a_1,\ a_2,\ a+r,\ -r \\ c_1,\ c_2,\ c_3 \end{matrix}\right], \quad F \equiv \frac{(a)_{r-3}}{r!}\,{}_4F_3\left[\begin{matrix} a_1,\ a_2,\ a+r-3,\ -r \\ c_1,\ c_2,\ c_3 \end{matrix}\right].$$

Then
$$F_t = (-)^{r-t}(\delta+a+r-3)_{3-r+t}(\delta-r)_{r-t}. \tag{27}$$

Now, substituting $x = 1$ in the differential equation that defines F, we see that F is annihilated by the operator

$$\delta(\delta+c_1-1)(\delta+c_2-1)(\delta+c_3-1)-$$
$$-(\delta+a_1)(\delta+a_2)(\delta+a_3+r-3)(\delta-r), \tag{28}$$

where, of course, the differentiations are to be performed *before* the substitution. This operator is of the third order and therefore can be rewritten in the form

$$A_r(\delta+a+r-3)_3+B_r(\delta+a+r-3)_2(\delta-r)+$$
$$+C_r(\delta+a+r-3)(\delta-r)_2+D_r(\delta-r)_3, \tag{29}$$

for suitable values of the constants $A_r,\dots, D_r$. Operating on F we then get, in virtue of (27),

$$A_r F_r - B_r F_{r-1} + C_r F_{r-2} - D_r F_{r-3} = 0,$$

a recurrence-formula for F_r. By comparing the two forms (28), (29) we can evaluate $A_r,\dots, D_r$; of these A_r, D_r are simple but B_r, C_r less so. However, we find without difficulty that, as $r \to \infty$,

$$A_r \sim B_r \sim -C_r \sim -D_r,$$

so that to a first approximation F_r satisfies the recurrence-formula

$$F_r - F_{r-1} - F_{r-2} + F_{r-3} = 0,$$

i.e.
$$F_r \sim \alpha+\beta r+(-1)^r\gamma$$

for some finite α, β, γ. Thus $F_r = O(r)$, which is sufficient for my purpose.

6.5. I shall not discuss the region of convergence of the double series (8)–(12), since this would require more preparatory work than their possible importance appears to justify.

For the five simple series whose coefficients involve terminated hypergeometric series we can now make use of the inequalities (78), (77), (85), (81) of (2) and give their intervals of convergence as

$$(1)\quad |x| < \tfrac{1}{2};\qquad (2)\quad |x| < 1;\qquad (17), (19)\quad |xy| < 1;$$
$$(18)\quad |xy| < \tfrac{1}{2}.$$

REFERENCES

1. W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge, 1935).
2. J. L. Burchnall and T. W. Chaundy, *Quart. J. of Math.* (Oxford), 11 (1940), 249–70.
3. ——— ——— ibid. 12 (1941), 112–28.
4. L. Saalschütz, *Zeits. für Math. u. Phys.* 36 (1891), 278–95.
5. F. J. W. Whipple, *Proc. London Math. Soc.* (2), 25 (1926), 525–44.
6. ——— ibid. 24 (1926), 247–63.
7. ——— ibid. 26 (1927), 257–72.

[*Added 1 September 1942*]. Professor J. L. Burchnall points out to me that the analysis of § 1 leads to the two theorems on reciprocal (or conjugate) transforms of series:

(A) *if* $$\phi_r = \sum_{s=0}^{r} \frac{(-r)_s}{s!}\theta_s \quad (r = 0, 1,...),$$

then $$\theta_r = \sum_{s=0}^{r} \frac{(-r)_s}{s!}\phi_s \quad (r = 0, 1,...);$$

(B) *if* $$\phi_r = \sum_{s=0}^{r} \frac{(-r)_s \theta_s}{s!\,(c+r+s-1)_{r-s}} \quad (r = 0, 1,...),$$

then $$\theta_r = \sum_{s=0}^{r} \frac{(-r)_s \phi_s}{s!\,(c+2s)_{r-s}} \quad (r = 0, 1,...).$$

He had previously remarked that the 'inverse pairs' of our expansions (2) (26)–(55) are examples of the analogous theorems on transforms:

(C) *if* $$\phi_r = \sum_{s=r}^{\infty} \frac{(-s)_r}{r!}\theta_s \quad (r = 0, 1,...),$$

then $$\theta_r = \sum_{s=r}^{\infty} \frac{(-s)_r}{r!}\phi_s \quad (r = 0, 1,...);$$

(D) $$\phi_r = \sum_{s=r}^{\infty} \frac{(-s)_r \theta_s}{r!\,(c+s-r-1)_{s-r}} \quad (r = 0, 1,...),$$

then $$\theta_r = \sum_{s=r}^{\infty} \frac{(-s)_r \phi_s}{r!\,(c)_{s-r}} \quad (r = 0, 1,...).$$

A SUMMATION FORMULA ASSOCIATED WITH FINITE TRIGONOMETRIC INTEGRALS

By D. G. KENDALL (*Oxford*)

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1. In this paper we consider the class of functions $f(x)$ which satisfy the summation formula

$$\tfrac{1}{2}\lambda f(0)+\lambda\sum_{n=1}^{\infty} f(n\lambda) = \int_0^{\infty} f(x)\,dx, \tag{1}$$

for all values of λ in some interval $0 < \lambda < \lambda_1$, i.e. the class of functions for which, with a sufficiently fine scale of subdivision, the trapezoidal rule gives an exact estimate of the infinite integral. With suitable restrictions we shall show that it is necessary and sufficient for the truth of (1) that $f(x)$ should have a Fourier cosine-transform $F_c(t)$ such that

$$F_c(t) = 0 \quad \text{for all } t > 2\pi/\lambda_1. \tag{2}$$

The formula (1) and its association with the vanishing of a Fourier transform do not appear to have been noticed before. In the theories associated with individual functions, however, there are many pairs of formulae of the type

$$\sum_{n=-\infty}^{\infty} f(n) = A = \int_{-\infty}^{\infty} f(x)\,dx,$$

customarily referred to as *integral-series analogues*, which are really particular cases of (1). Thus there is the striking analogy between the formulae of Hansen,*

$$\sum_{r=-\infty}^{\infty} J_{m+r}(z)J_{n-r}(z) = J_{m+n}(2z), \tag{3}$$

and Ramanujan,†

$$\int_{-\infty}^{\infty} J_{\mu+t}(z)J_{\nu-t}(z)\,dt = J_{\mu+\nu}(2z), \quad \text{if } \mathbf{R}(1+\mu+\nu) > 0. \tag{4}$$

In § 5 we shall obtain the complete result which makes clear the relationship of (3) and (4); it is

$$\int_{-\infty}^{\infty} J_{\mu+t}(z)J_{\nu-t}(z)\,dt = J_{\mu+\nu}(2z) = \lambda\sum_{n=-\infty}^{\infty} J_{\mu+n\lambda}(z)J_{\nu-n\lambda}(z), \tag{5}$$

if $\mathbf{R}(1+\mu+\nu) > 0$ and $0 < \lambda < 2$.

* Watson, (7), 30.

† Ibid. 449.

2. It is easy to show in a formal way that (1) and (2) are equivalent. In the first place, defining the Fourier cosine-transform of $f(x)$ as

$$F_c(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(x)\cos xt\, dx,$$

we have the Poisson summation-formula,

$$\surd\alpha\Big\{\tfrac{1}{2}f(0)+\sum_{n=1}^{\infty} f(n\alpha)\Big\} = \surd\beta\Big\{\tfrac{1}{2}F_c(0)+\sum_{n=1}^{\infty} F_c(n\beta)\Big\},$$

where $\alpha\beta = 2\pi$ and $\alpha > 0$. We can write this in the form

$$R(\alpha) \equiv \Big\{\tfrac{1}{2}\alpha f(0)+\alpha\sum_{n=1}^{\infty} f(n\alpha)-\int_0^\infty f(x)\,dx\Big\} = \sqrt{(2\pi)}\sum_{n=1}^{\infty} F_c(n\beta).$$

On the other hand, if $\mu(m)$ is the Möbius symbol, so that

$$\sum_{m|n}\mu(m) = \begin{cases} 0 & (n \neq 1), \\ 1 & (n = 1), \end{cases}$$

then

$$\sum_{m=1}^{\infty}\mu(m)R\left(\frac{\alpha}{m}\right) = \sqrt{(2\pi)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mu(m)F_c(mn\beta)$$
$$= \sqrt{(2\pi)}F_c(\beta).$$

Now (1) may be written

$$R(\alpha) = 0 \quad \text{for } 0 < \alpha < \lambda_1,$$

and clearly this is formally equivalent to

$$F_c(\beta) = 0 \quad \text{for } \beta > 2\pi/\lambda_1.$$

The development of the paper is as follows. We first obtain a number of theorems which state, under various conditions, that (2) implies (1). Some applications of these theorems are then discussed; of the examples given, the equations (5), (15), (16) are believed to be new. Next we obtain a symmetrical theorem which for a certain class of functions $f(x)$ states that (1) implies (2) and vice versa. Finally we show how, in a certain sense, the approximate truth of (1) is equivalent to the approximate truth of (2).

3. In this section we state and prove the following theorem.

THEOREM 1. *Let $\psi(t)$ be a function of $L(0,a)$ and let*

$$f(x) = \int_0^a \psi(t)\cos xt\, dt.$$

Then, for all λ in the range $0 < \lambda < 2\pi/a$,

$$\tfrac{1}{2}\lambda f(0)+\lambda\sum_{n=1}^{\infty} f(n\lambda) = \int\limits_0^{\infty} f(x)\,dx, \tag{1}$$

if either side converges.

In particular, if $\psi(|t|)$ satisfies a condition sufficient for the convergence of its Fourier series at the point $t = 0$ to the sum S, then (1) *is true, and the common value of either side is $\frac{1}{2}\pi S$.*

The basis of the proof of Theorem 1 is the complete equivalence of the necessary and sufficient conditions for the Fourier series and Fourier integral representations of an integrable function to converge at a given point and to a given sum. While this fact is well known, it does not appear to be stated explicitly in the text-books; we therefore include the following proof. By the usual discussion* of the convergence problems for the two representations we need only establish the equivalence of the two statements

$$\lim_{X\to\infty}\int\limits_0^{\delta}\psi(t)\frac{\sin Xt}{t}\,dt = C, \tag{6}$$

$$\lim_{N\to\infty}\int\limits_0^{\delta}\psi(t)\frac{\sin(N+\frac{1}{2})\lambda t}{t}\,dt = C, \tag{7}$$

for some $\delta > 0$, when $\psi(t)$ is a function of $L(0,\delta)$, and when X and N tend to infinity through continuous and integer values respectively. We write†

$$J_X = \int\limits_0^{\delta}\psi(t)\{\sin Xt-\sin(N+\tfrac{1}{2})\lambda t\}\,\frac{dt}{t}$$

$$= 2\int\limits_0^{\delta}\psi(t)\sin\tfrac{1}{2}\{X-(N+\tfrac{1}{2})\lambda\}t\cos\tfrac{1}{2}\{X+(N+\tfrac{1}{2})\lambda\}t\,\frac{dt}{t},$$

and we make N equal to the integer part of $(X/\lambda-\frac{1}{2})$. Then, for sufficiently small δ,
$$\frac{\sin\frac{1}{2}\{X-(N+\frac{1}{2})\lambda\}t}{t}$$

is a steadily decreasing positive function of t in $0 \leqslant t \leqslant \delta$, and so,

* See e.g. Titchmarsh, (**5**), ch. xiii, and (**6**), ch. i.

† For the proof that (7) implies (6) I am indebted to Mr. F. M. C. Goodspeed.

by the second mean-value theorem for integrals, there is a θ (depending on X) such that $0 \leqslant \theta \leqslant \delta$, and

$$J_X = \{X-(N+\tfrac{1}{2})\lambda\} \int_0^\theta \psi(t)\cos \tfrac{1}{2}\{X+(N+\tfrac{1}{2})\lambda\}t\, dt.$$

A simple extension of the Riemann-Lebesgue theorem then shows that $J_X \to 0$ as $X \to \infty$, and so (6) and (7) are equivalent.

To prove Theorem 1 we suppose that $\psi(t)$, $f(x)$, and λ are defined as in the enunciation, and we define $\phi(t)$ by

$$\phi(t) = \begin{cases} \psi(t) & (0 \leqslant t \leqslant a), \\ 0 & (a < t < 2\pi/\lambda), \end{cases}$$

$$\phi(t) = \phi(t+2\pi/\lambda).$$

The Fourier series for $\phi(t)$ in the interval $(0, 2\pi/\lambda)$ is

$$\phi(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\lambda t + b_n \sin n\lambda t\},$$

where $$a_n = \frac{\lambda}{\pi} \int_0^{2\pi/\lambda} \phi(t)\cos n\lambda t\, dt = \frac{\lambda}{\pi} f(n\lambda).$$

The function $\chi(t)$, defined by

$$\chi(t) = \psi(t) \quad (0 \leqslant t \leqslant a),$$

$$\chi(t) = 0 \quad \text{elsewhere},$$

has the Fourier transform

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \chi(t)e^{ixt}\, dt = \frac{1}{\sqrt{(2\pi)}}\{f(x)+ig(x)\},$$

where $g(x)$ is an odd function of x. In the neighbourhood of $t = 0$, $\phi(t)$ and $\chi(t)$ coincide, and so the necessary and sufficient conditions for the convergence to a given sum of the two representations

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{\lambda}{2\pi} f(0) + \frac{\lambda}{\pi} \sum_{n=1}^{\infty} f(n\lambda) \tag{8}$$

and $$\lim_{X\to\infty} \frac{1}{2\pi} \int_{-X}^{X} \{f(x)+ig(x)\}\, dx = \frac{1}{\pi} \int_0^{\infty} f(x)\, dx \tag{9}$$

are completely equivalent. This proves the first part of the theorem. To prove the second part we observe that, for a sufficiently small δ,

$$\phi(t) = \chi(t) = \psi(t) \quad (0 \leqslant t \leqslant \delta),$$
$$\phi(t) = \chi(t) = 0 \quad (-\delta \leqslant t < 0).$$

Thus, if $\psi(|t|)$ satisfies a condition sufficient for the convergence of its Fourier series at $t = 0$ to the sum S, the series (8) will converge to the sum $\frac{1}{2}S$.

In this proof of Theorem 1 we have implicitly assumed that $\psi(t)$ and $f(x)$ are real-valued functions, but the restriction is unnecessary. If $\psi(t)$ and $f(x)$ are complex-valued, we apply the theorem separately to their real and imaginary parts, and combine the results.

We will now obtain the following more general result.

THEOREM 2. *Let $\psi(t)$, $f(x)$ be defined as in Theorem* 1. *Then for any κ (real or complex), and for all $0 < \lambda < 2\pi/a$,*

$$\lambda \sum_{n=-\infty}^{\infty} f(\kappa+n\lambda) = \int_{-\infty}^{\infty} f(x)\,dx, \tag{10}$$

if either side converges.

In particular, if $\psi(|t|)$ satisfies a condition sufficient for the convergence of its Fourier series at $t = 0$ to the sum S, then (10) *is true, and the common value of either side is πS.*

Proof. (i) If $\psi(t)$ satisfies the condition stated at the end of Theorem 2, then so does $\psi(t)\cos\kappa t$. For

$$\lim_{N\to\infty} \int_0^{\delta} \psi(t)\frac{1-\cos\kappa t}{t}\sin(N+\tfrac{1}{2})\lambda t\,dt = 0.$$

Also
$$f(\kappa+x)+f(\kappa-x) = \int_0^{a} \{2\psi(t)\cos\kappa t\}\cos xt\,dt,$$

and so, by applying Theorem 1, any one of the three alternative conditions of Theorem 2 implies that

$$\lim_{N\to\infty} \lambda \sum_{n=-N}^{N} f(\kappa+n\lambda) = \lim_{X\to\infty} \int_{-X+\kappa}^{X+\kappa} f(x)\,dx \quad (= \pi S).$$

It is necessary to prove that the series and the integral converge in the ordinary sense and not merely as Cauchy principal values.

(ii) We have
$$\int_{-\infty+\kappa}^{\infty+\kappa} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx, \tag{11}$$
if either side converges. For
$$\lim_{X\to\infty}\int_{X}^{X+\kappa} f(x)\,dx$$
$$= \lim_{X\to\infty} 2\int_0^a \psi(t)\frac{\sin\frac{1}{2}\kappa t}{t}\{\cos\tfrac{1}{2}\kappa t\cos Xt-\sin\tfrac{1}{2}\kappa t\sin Xt\}\,dt,$$
and this vanishes, by the Riemann-Lebesgue theorem. The result now follows on using Cauchy's theorem, and recalling that $f(x)$ is an even integral function.

(iii) If $0<\lambda<2\pi/a$, the two series
$$\sum_{n=-\infty}^{\infty} f(\kappa+n\lambda), \qquad \sum_{n=-\infty}^{\infty} f(n\lambda)$$
converge or diverge together. For
$$\sum_{n=1}^{N}\{f(n\lambda)-f(\kappa+n\lambda)\} = \int_0^a \psi(t)\frac{\sin\kappa t\cos\frac{1}{2}\lambda t-(1-\cos\kappa t)\sin\frac{1}{2}\lambda t}{2\sin\frac{1}{2}\lambda t}\,dt+$$
$$+\int_0^a \psi(t)\frac{1-\cos\kappa t}{2\sin\frac{1}{2}\lambda t}\sin(N+\tfrac{1}{2})\lambda t\,dt-$$
$$-\int_0^a \psi(t)\frac{\sin\kappa t}{2\sin\frac{1}{2}\lambda t}\cos(N+\tfrac{1}{2})\lambda t\,dt.$$
The first term is finite and independent of N, while the last two terms tend to zero as $N\to\infty$. Similarly, the two series converge or diverge together at the lower limits.

Combining (i), (ii), and (iii) with Theorem 1 we obtain Theorem 2. A number of examples of the theorems will be given in § 5. In most of these $\psi(|t|)$ is differentiable at $t=0$, and so by Dini's test* its Fourier series converges at $t=0$ to the sum $\psi(0)$.

4. The following result, although less powerful than Theorem 1, has the advantage that the conditions are imposed primarily on $f(x)$.

* Titchmarsh, (5), 406.

THEOREM 3. *Let $f(x)$ be a function of $L^p(0,\infty)$ $(1 \leqslant p \leqslant 2)$ such that its Fourier cosine-transform $F_c(t)$ vanishes for almost all $t > a$. Then there is a continuous function $f_0(x)$, equal almost everywhere to $f(x)$, such that for $0 < \lambda < 2\pi/a$,*

$$\tfrac{1}{2}\lambda f_0(0)+\lambda \sum_{n=1}^{\infty} f_0(n\lambda) = \int_0^{\infty} f_0(x)\,dx,$$

if either side converges.

If $p = 1$, $F_c(t)$ is continuous and vanishes for $t \geqslant a$. If $1 < p \leqslant 2$, $F_c(t)$ is $L^{p'}(0,\infty)$ (where $p' = p/(p-1) \geqslant 2$) and vanishes for almost all $t > a$. Thus in either case $F_c(t)$ is $L(0,\infty)$. Also*

$$f(x) = (C,1)\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty} F_c(t)\cos xt\,dt,$$

for almost all x, and so the continuous function

$$f_0(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^{a} F_c(t)\cos xt\,dt$$

exists for all x and is equal to $f(x)$ almost everywhere. The result now follows on applying Theorem 1 to $f_0(x)$.

We next obtain a theorem in which all the conditions are imposed on $f(x)$, and no mention is made of its Fourier transform. We quote for convenience the following fundamental result of Paley and Wiener.†

The following two classes of functions are identical:

(i) *the class of integral functions $f(z)$ which are L^2 on the real axis and which, for all z and for some positive a, satisfy the condition*

$$f(z) = O(e^{a|z|});$$

(ii) *the class of functions $f(z)$ defined by*

$$f(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a} \phi(t)e^{-izt}\,dt,$$

where $\phi(t)$ is $L^2(-a,a)$.

* Titchmarsh, (**6**), Theorems 14 and 59, and the generalization of the latter for the class L^p.

† Paley and Wiener, (**4**), Theorem X.

Following Hardy,* we will refer to functions of either class as Paley-Wiener functions (of type a).

THEOREM 4. *Let $f(z)$ be an even Paley-Wiener function of type a. Then, for $0 < \lambda < 2\pi/a$,*

$$\tfrac{1}{2}\lambda f(0)+\lambda \sum_{n=1}^{\infty} f(n\lambda) = \int_0^{\infty} f(x)\,dx,$$

if either side converges.

For $f(x)$ is $L^2(0,\infty)$, and has a Fourier cosine-transform $\phi(t)$ which vanishes for almost all $t > a$. The result follows on using Theorem 3 and noting that $f(x)$ is continuous.

A similar but more powerful result can be obtained by complex-variable methods. Theorem 5 is due to Professor E. C. Titchmarsh, and I am very grateful to him for permission to include it here.

THEOREM 5. *Let the integral function $f(z)$ be $O(e^{a|z|})$ as $|z| \to \infty$, and on the real axis let $f(x) \to 0$ as $|x| \to \infty$. Let $0 < \lambda < 2\pi/a$. Then*

$$\lim_{N\to\infty}\left\{\lambda \sum_{n=-N}^{N} f(n\lambda) - \int_{-(N+\frac{1}{2})\lambda}^{(N+\frac{1}{2})\lambda} f(x)\,dx\right\} = 0. \tag{12}$$

If Γ_N denotes the square contour with vertices at the points $(N+\frac{1}{2})\lambda(\pm 1 \pm i)$, we have

$$\frac{1}{2\pi i}\int_N \pi \cot\frac{\pi z}{\lambda} f(z)\,dz = \lambda \sum_{n=-N}^{N} f(n\lambda). \tag{13}$$

Let us write $$\pi\cot\frac{\pi z}{\lambda} = \pm \pi i \pm \frac{2\pi i}{e^{\pm 2\pi i z/\lambda}-1},$$

in the lower and upper half-planes respectively. The contour integral on the left of (13) then splits up into four terms, and the two containing $\pm\pi i$ combine to give

$$\int_{-(N+\frac{1}{2})\lambda}^{(N+\frac{1}{2})\lambda} f(x)\,dx,$$

when the paths of integration are deformed into the real axis. We will prove (12) by showing that the two remaining terms tend to zero as $N \to \infty$.

We must first show that $f(z) = o(e^{a|y|})$ (where $z = x+iy$), uniformly as $|z| \to \infty$. For let $g(z) = e^{iaz}f(z)$; this is an integral function of order 1, bounded on the positive imaginary axis and tending to

* Hardy, (2).

zero as $z \to \infty$ in either direction along the real axis. It follows from standard Phragmén-Lindelöf theorems* that $g(z) \to 0$ as $z \to \infty$ uniformly in the upper half-plane; i.e. $f(z) = o(e^{ay})$ uniformly in $y \geqslant 0$. A similar argument shows that $f(z) = o(e^{-ay})$ uniformly in $y \leqslant 0$.

Of the remaining contributions to the left-hand side of (13), each of the four integrals along paths parallel to the imaginary axis is of order

$$o\left\{\int_0^{(N+\frac{1}{2})\lambda} \frac{e^{ay}\,dy}{e^{2\pi y/\lambda}+1}\right\};$$

while each of the two integrals along paths parallel to the real axis is of order

$$o\left\{\int_{-(N+\frac{1}{2})\lambda}^{(N+\frac{1}{2})\lambda} \frac{e^{a\lambda(N+\frac{1}{2})}\,dx}{e^{2\pi(N+\frac{1}{2})}-1}\right\}.$$

Both expressions tend to zero as $N \to \infty$; this completes the proof of Theorem 5.

5. Examples. The following examples are all covered by Theorems 1 and 2.

(i) Since
$$\frac{\sin x}{x} = \int_0^1 \cos xt\,dt,$$
we have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi = \tfrac{1}{2}\lambda + \sum_{n=1}^\infty \frac{\sin n\lambda}{n} \quad (0 < \lambda < 2\pi).$$

More generally, for any positive integer m,

$$\left(\frac{\sin x}{x}\right)^m = \int_0^m \psi(t)\cos xt\,dt,$$

where $\psi(t)$ is $L(0,m)$, and so

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^m dx = \tfrac{1}{2}\lambda + \lambda^{1-m}\sum_{n=1}^\infty \left(\frac{\sin n\lambda}{n}\right)^m \quad (0 < \lambda < 2\pi/m).$$

(ii) When $\mathbf{R}(\nu+\frac{1}{2}) > 0$, we have†

$$x^{-\nu}J_\nu(x) = \frac{2^{1-\nu}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos xt\,dt.$$

* Titchmarsh, (5), 5.61 and 5.63. † Watson, (7), 48.

With this condition on ν, Theorem 1 applies, for then $(1-t^2)^{\nu-\frac{1}{2}}$ is $L(0,1)$ and differentiable at $t=0$. Thus

$$\int_0^\infty x^{-\nu}J_\nu(x)\,dx = \frac{2^{-\nu}\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})} = \frac{\lambda 2^{-1-\nu}}{\Gamma(1+\nu)}+\lambda^{1-\nu}\sum_{n=1}^{\infty} n^{-\nu}J_\nu(n\lambda),$$

if $\mathbf{R}(\nu+\frac{1}{2})>0$ and $0<\lambda<2\pi$. In particular,

$$\int_0^\infty J_0(x)\,dx = 1 = \tfrac{1}{2}\lambda+\lambda\sum_{n=1}^{\infty} J_0(n\lambda) \quad (0<\lambda<2\pi).$$

(iii) The formula

$$\frac{2^{-\mu-\nu}\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos^{\mu+\nu}\theta\cos(\mu-\nu)\theta\,d\theta,$$

due to Cauchy,* is valid for $\mathbf{R}(\mu+\nu+1)>0$. It follows that

$$\frac{2^{-\mu-\nu}\Gamma(\mu+\nu+1)}{\Gamma(1+\frac{1}{2}\mu+\frac{1}{2}\nu+x)\Gamma(1+\frac{1}{2}\mu+\frac{1}{2}\nu-x)} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos^{\mu+\nu}\theta\cos 2x\theta\,d\theta.$$

We now apply Theorem 2 and formula (11), with $\kappa=\frac{1}{2}(\mu-\nu)$, and obtain

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(1+\mu+x)\Gamma(1+\nu-x)}$$
$$= \frac{2^{\mu+\nu}}{\Gamma(1+\mu+\nu)} = \lambda\sum_{n=-\infty}^{\infty} \frac{1}{\Gamma(1+\mu+n\lambda)\Gamma(1+\nu-n\lambda)}, \quad (14)$$

if $\mathbf{R}(\mu+\nu+1)>0$ and $0<\lambda<2$. The evaluation of the integral in (14) is due to Ramanujan; the series appears to have been known to Cauchy. When $\lambda=1$ we obtain

$$\sum_{r=0}^{n} {}^nC_r = 2^n.$$

(iv) In a similar manner we obtain the formulae (3), (4), (5) of § 1. Since†

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\,d\theta,$$

if $\mathbf{R}(\mu+\nu+1)>0$, we have

$$J_{\frac{1}{2}\mu+\frac{1}{2}\nu+x}(z)J_{\frac{1}{2}\mu+\frac{1}{2}\nu-x}(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2z\cos\theta)\cos 2x\theta\,d\theta.$$

* Watson, (7), 449.

† Ibid. 150.

As in (iii) we apply Theorem 2 and (11), with $\kappa = \frac{1}{2}(\mu-\nu)$, and obtain (5). We have of course assumed that z is not zero. Formally, when z approaches zero, (5) reduces to (14).

(v) The Dirichlet-Mehler integral* for the Legendre function of the first kind can be written in the form

$$P_{\nu-\frac{1}{2}}(\cos\theta) = \frac{\sqrt{2}}{\pi}\int\limits_0^\theta \frac{\cos\nu t\,dt}{(\cos t-\cos\theta)^{\frac{1}{2}}}.$$

This is valid for all real ν and for $0 < \theta < \pi$. Theorem 1 applies, since $(\cos t-\cos\theta)^{-\frac{1}{2}}$ is $L(0,\theta)$ and is differentiable at $t = 0$. Thus

$$\int\limits_0^\infty P_{\nu-\frac{1}{2}}(\cos\theta)\,d\nu = \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}\theta = \tfrac{1}{2}\lambda P_{-\frac{1}{2}}(\cos\theta)+\lambda\sum_{n=1}^\infty P_{n\lambda-\frac{1}{2}}(\cos\theta). \qquad (15)$$

Theorem 2, with $\kappa = \frac{1}{2}$, gives

$$\int\limits_{-\infty}^\infty P_\nu(\cos\theta)\,d\nu = \operatorname{cosec}\tfrac{1}{2}\theta = \lambda\sum_{n=-\infty}^\infty P_{n\lambda}(\cos\theta). \qquad (16)$$

Both (15) and (16) hold when $0 < \theta < \pi$ and $0 < \lambda < 2\pi/\theta$.

6. In this section we make rigorous the analysis of § 2, and prove the equivalence of (1) and (2) for a certain class of functions $f(x)$.

THEOREM 6. *Let $f(x)$ be differentiable and tend to zero as $x \to \infty$, and let $f(x)$, $f'(x)$ both be of bounded variation in* $(0,\infty)$. *Then the following statements are equivalent:*

(i) $$\lim_{N\to\infty}\left\{\tfrac{1}{2}\lambda f(0)+\lambda\sum_{n=1}^N f(n\lambda)-\int\limits_0^{(N+\frac{1}{2})\lambda} f(x)\,dx\right\} = 0,$$

for all $0 < \lambda < 2\pi/a$;

(ii) $F_c(t) = 0$, *for all* $t > a$.

We use a theorem due to Ferrar and Titchmarsh,† which under conditions satisfied here states that

$$\lim_{N\to\infty}\left\{\tfrac{1}{2}\lambda f(0)+\lambda\sum_{n=1}^N f(n\lambda)-\int\limits_0^{(N+\frac{1}{2})\lambda} f(x)\,dx\right\} = \sqrt{(2\pi)}\sum_{n=1}^\infty F_c\left(\frac{2n\pi}{\lambda}\right). \qquad (17)$$

Obviously (ii) implies (i). To prove the converse statement suppose that (i) is true, so that

$$\sum_{n=1}^\infty F_c(nt) = 0,$$

* Hobson, (**3**), 267.

† Titchmarsh, (**6**), Theorem 45.

for all $t > a$. As in § 2 we then have

$$\sum_{m=1}^{\infty} \mu(m) \sum_{n=1}^{\infty} F_c(mnt) = 0 \quad (t > a), \tag{18}$$

and (ii) follows on rearrangement of the double series. We shall justify this by showing that the double series is absolutely convergent.

The Fourier cosine-transform of $f(x)$ is here defined by the Cauchy integral,

$$F_c(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\to\infty} f(x) \cos xt\, dx,$$

which exists for all $t > 0$. We can integrate by parts, because $f(x)$ has a bounded derivative, and so

$$F_c(t) = -\frac{1}{t} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\to\infty} f'(x) \sin xt\, dx. \tag{19}$$

Now $f'(x)$ is of bounded variation in $(0, \infty)$, and the integral in (19) exists for all $t > 0$; we must therefore have $f'(x) \to 0$ as $x \to \infty$. It follows that we can write

$$f'(x) = g_1(x) - g_2(x),$$

where both $g_1(x)$ and $g_2(x)$ are bounded and decrease steadily to zero as $x \to \infty$. On substituting in (19), and using the second mean-value theorem for integrals, we obtain

$$F_c(t) = O(1/t^2), \quad \text{as } t \to \infty.$$

The double series (18) is dominated by

$$\sum_{n=1}^{\infty} d(n)\, |F_c(nt)|,$$

where $d(n)$ is the number of divisors of n, and this series converges because

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^2}$$

is convergent. Thus the rearrangement of (18) is justified, and the theorem is proved.

7. A generalization of the results of the last section throws an interesting light on certain numerical approximations noticed by Aitken.*

* Aitken, (**1**), 45.

Let $f(x)$ satisfy the general conditions of Theorem 6, and write $R(\lambda)$ for the left-hand side of (17). Then it is easily seen that, when $\alpha > 1$,

$$|F_c(t)| \leqslant At^{-\alpha} \quad \text{for all } t > a$$

implies that

$$|R(\lambda)| \leqslant A\zeta(\alpha)(2\pi)^{\frac{1}{2}-\alpha}\lambda^{\alpha} \quad \text{for all } 0 < \lambda < 2\pi/a.$$

Similarly, $$|R(\lambda)| \leqslant B\lambda^{\alpha} \quad \text{for all } 0 < \lambda < 2\pi/a$$

implies that

$$|F_c(t)| \leqslant B\zeta(\alpha)(2\pi)^{\alpha-\frac{1}{2}}t^{-\alpha} \quad \text{for all } t > a.$$

Here $\zeta(\alpha)$ is the Riemann zeta function.

Thus the 'approximate' truth of (1) is equivalent to the 'approximate' truth of (2). Aitken compared the values of $R(1)$ for the two functions

$$f(x) = e^{-\frac{1}{2}x^2} \quad \text{and} \quad f(x) = 1/(1+x^2).$$

In the first case $R(1)$ is exceedingly small, while in the second it is much larger. We can see now that these facts are intimately connected with the relative smallness at infinity of the corresponding Fourier cosine-transforms, which are

$$F_c(t) = e^{-\frac{1}{2}t^2} \quad \text{and} \quad F_c(t) = \sqrt{(\tfrac{1}{2}\pi)}e^{-t}.$$

In conclusion I wish to express my sincere thanks to Professor E. C. Titchmarsh for his kindness in reading a first draft of this paper and making a number of valuable suggestions, and also for permission to include Theorem 5.

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REFERENCES

1. A. C. Aitken, *Statistical Mathematics* (Edinburgh, 1939).
2. G. H. Hardy, *Proc. Cambridge Phil. Soc.* 37 (1941), 332.
3. E. W. Hobson, *Theory of Spherical and Ellipsoidal Harmonics* (Cambridge, 1931).
4. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (New York, 1934).
5. E. C. Titchmarsh, *Theory of Functions* (Oxford, 1932).
6. —— *Theory of Fourier Integrals* (Oxford, 1937).
7. G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922).

ON NEIGHBOURS OF HIGHER DEGREE IN FAREY SERIES

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LET $\mathfrak{F}_n$ denote the Farey series of order n, that is, the ascending series of irreducible fractions between (and including) $\frac{0}{1}$ and $\frac{1}{1}$ whose denominators do not exceed n, and let

$$\frac{a}{b}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k}, \ldots, \frac{a_m}{b_m}, \ldots$$

be successive fractions in $\mathfrak{F}_n$, in their lowest terms.

The elements a/b and a_k/b_k of $\mathfrak{F}_n$ may be called *kth neighbours* in $\mathfrak{F}_n$, and we shall say that they are *similarly ordered** if

$$(a_k-a)(b_k-b) \geqslant 0.$$

In a recent paper of mine it was essential for the argument that first and second neighbours in any Farey series should be similarly ordered.† As a corollary, I proved the similar ordering of third neighbours, with the exception of $\mathfrak{F}_4$. It is the same with fourth neighbours, though there are more exceptions. Professor G. H. Hardy subsequently suggested it might be true, in general, that kth neighbours are similarly ordered, with a finite number (dependent on k) of exceptions.

The present Note contains an elementary proof of this theorem (§ 1). The resulting limit for exceptions is, numerically, very weak. Therefore the cases $k \leqslant 5$ are studied in detail (§ 2) by means of a method practicable for small values of k.

1. We begin by proving a lemma.

LEMMA. *Given a number k, there is a number $L = L(k)$ such that any open interval whose length is not less than L contains k co-prime numbers.*

Plainly, we may put $L(1) = 2$. Assume then that $k > 1$, and let P_k denote the product $p_1 p_2 \ldots p_k$ of the first k primes ($p_1 = 2$). We shall find that $L = P_k + 1$ satisfies the Lemma.

* In accordance with G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge, 1934), 43.

† A. E. Mayer, 'A mean-value theorem concerning Farey series': *Quart. J. of Math.* (Oxford), 13 (1942), 48–57.

Any interval S, not shorter than P_k+1, contains in its interior a multiple of P_k, say λP_k, and also $\lambda P_k+\epsilon p_k$, where ϵ is either $+1$ or -1; for one of the two distances between λP_k and the boundaries of S is at least $\frac{1}{2}(P_k+1) > \frac{1}{2}P_k \geqslant p_k$.

Thus the k numbers

$$\lambda P_k+\epsilon p_1, \qquad \lambda P_k+\epsilon p_2, \qquad \ldots, \qquad \lambda P_k+\epsilon p_k$$

lie in S. It is easy to see that they are co-prime. Suppose that

$$1 \leqslant i < j \leqslant k,$$

and
$$d \mid (\lambda P_k+\epsilon p_i), \qquad d \mid (\lambda P_k+\epsilon p_j).$$

Hence
$$d \mid (p_j-p_i),$$

and so $d < p_j$. Therefore any prime factor of d would be less than p_j. On the other hand, the primes $p_1, p_2, \ldots, p_{j-1}$ do not divide $\lambda P_k+\epsilon p_j$. It follows that $d = 1$.

Our main object is to prove

THEOREM 1. *For any number k a number $N = N(k)$ exists such that kth neighbours in $\mathfrak{F}_n$ are similarly ordered, if only $n \geqslant N$.*

Suppose that a/b and a_m/b_m are not similarly ordered, i.e. that

$$(a_m-a)(b_m-b) < 0. \tag{1}$$

Since $a/b < a_m/b_m$, it is obvious that

$$a < a_m, \qquad b > b_m. \tag{2}$$

From $b_m > 0$ and (2) we deduce, in turn, that $b > 1$, $a > 0$, $a_m > 1$, and $0 < a/b < a_m/b_m < 1$.

Instead of (2) we may write

$$a+1 \leqslant a_m, \qquad b-1 \geqslant b_m;$$

and so
$$0 < \frac{a}{b} < \frac{a+1}{b-1} \leqslant \frac{a_m}{b_m} < 1. \tag{3}$$

The proof of Theorem 1 requires a number N such that between a/b and a_m/b_m there are at least k different fractions whose denominators do not exceed n, if $n \geqslant N$. We shall determine N so that k different fractions with denominators not greater than n lie between a/b and $(a+1)/(b-1)$; then, *a fortiori*, by (3), they lie between a/b and a_m/b_m. This may be done, in the first place, for the special case $b = n$, which is covered by the following theorem.

THEOREM 2. *If $0 < a/b < (a+1)/(b-1) < 1$ and $b \geqslant kL$, then k different fractions whose denominators do not exceed b lie between a/b and $(a+1)/(b-1)$.*

This holds for $k = 1$, since $a/b < (a+1)/b < (a+1)/(b-1)$; and we may therefore assume that $k > 1$.

We have to distinguish two cases. Suppose first that

$$\frac{a+1}{b-1} < \frac{1}{k-1}.$$

If so, $$\frac{a+1}{b-1} = a\Big/\left(b-1-\frac{b-1}{a+1}\right) > \frac{a}{b-k}.$$

Then $$\frac{a}{b} < \frac{a}{b-1} < \frac{a}{b-2} < \ldots < \frac{a}{b-k} < \frac{a+1}{b-1},$$

and Theorem 2 follows. It will be observed that $b > k$, by hypothesis.

Consider next the alternative $\dfrac{a+1}{b-1} \geqslant \dfrac{1}{k-1}$, that is,

$$a \geqslant \frac{b-1}{k-1}-1. \tag{4}$$

If $$x\frac{a+1}{b-1}-x\frac{a}{b} > 1, \tag{5}$$

there is an integer y such that

$$x\frac{a}{b} < y < x\frac{a+1}{b-1},$$

i.e. $$\frac{a}{b} < \frac{y}{x} < \frac{a+1}{b-1}. \tag{6}$$

Now (5) is equivalent to

$$x > \frac{b(b-1)}{a+b},$$

and it follows from (4) that

$$\frac{b(b-1)}{a+b} \leqslant b(b-1)\Big/\left(\frac{b-1}{k-1}-1+b\right) = \frac{k-1}{k}b.$$

Hence (5) is certainly true if

$$\frac{k-1}{k}b < x < b.$$

The length of this interval is $b/k \geqslant L$. According to the Lemma it contains k co-prime numbers $x_1, x_2,\ldots, x_k$. There is a set $y_1, y_2,\ldots, y_k$

of corresponding integers such that the fractions y/x satisfy (6). To complete the proof of Theorem 2 it remains to show that these fractions have different values. Suppose the contrary, say

$$y_i/x_i = r/s = y_j/x_j \quad (i \neq j),$$

and so $\qquad y_i = \lambda r, \qquad y_j = \mu r, \qquad x_i = \lambda s, \qquad x_j = \mu s,$

where all letters denote positive integers. The highest common divisor of the denominators is

$$(x_i, x_j) = (\lambda s, \mu s) = s(\lambda, \mu) \geqslant s.$$

Since $0 < r/s < 1$, by (3) and (6), we have $s > 1$, and $(x_i, x_j) > 1$: a contradiction.

It is now easy to prove Theorem 1. We may take $N = k(k+1)L$. If $b \geqslant kL$, then Theorem 2 is all we need; if $b < kL$, then plainly

$$\frac{a}{b} < \frac{a+(a+1)}{b+(b-1)} < \frac{a+2(a+1)}{b+2(b-1)} < \ldots < \frac{a+k(a+1)}{b+k(b-1)} < \frac{a+1}{b-1},$$

and the largest denominator is

$$b+k(b-1) < (k+1)b < k(k+1)L.$$

It does not exceed n, if $n \geqslant N = k(k+1)L$.

2. In $k(k+1)(P_k+1)$ only a very large limit for N has been obtained. It yields, for instance, $N(4) = 4220$; while 10 is the minimum of $N(4)$, as we shall see. Moreover, $\mathfrak{F}_{4219}$ has about five million terms;* this makes it evident that the argument so far does not enable us, even when k is small, to find all not similarly ordered kth neighbours and thus to reach the lower bound of $N(k)$. The problem is, naturally, rather tiresome, and I content myself with solving it for $k \leqslant 5$.

The further argument is geometrical, and based on the representation of the terms a/b,... of $\mathfrak{F}_n$ by the points (b, a),... of the integral lattice (x, y). Let O denote the origin, and P, Q be the points representing a/b and $(a+1)/(b-1)$, respectively, where a and b satisfy (1).

This representation can be reversed, in a way, by applying the parallel displacement $x \to x-b$, $y \to y-a$. Then the coordinates of P become $(0, 0)$, and Q coincides with $(-1, 1)$. Into which lattice-point has O been moved?

* Cf. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford, 1938), 266.

The gradient of OP is positive, that of OQ less than unity, as (3) shows. Hence O lies inside the angular region Λ defined by

$$y < 0, \qquad y-x > 2.$$

We require the nine lattice-points

$$A = (0,1), \qquad B_i = (-i,0), \qquad C_i = (-i-1,-1) \quad (i = 1, 2, 3, 4),$$

which we arrange into four quintuples, namely

$$\alpha = \{A, B_1, B_2, B_3, B_4\}, \qquad \beta = \{A, B_1, B_2, B_3, C_4\},$$
$$\gamma = \{A, B_1, B_2, C_2, C_3\}, \qquad \delta = \{A, B_1, B_2, C_1, C_2\}.$$

The smallest convex polygon containing α excludes P and Q (Fig. 1). Hence there are straight lines that pass through P and

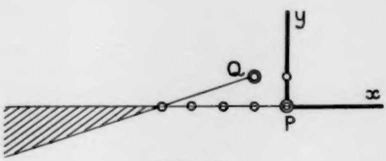

FIG. 1

Q, respectively, and do not dissect the polygon, such as the lines PB_i and QB_4. These are, in part, boundaries of the region Λ_α, where

$$y < 0, \qquad 3y-x > 4.$$

In what follows, P_α stands for any point of the set α.

If O were a point inside Λ_α, then the gradient of OP_α would be between those of OP and OQ. In other words, the value of the fraction represented by P_α would lie between a/b and $(a+1)/(b-1)$. It will be noticed that the abscissa of P_α is non-positive: P_α represents a fraction whose denominator does not exceed b. The reduced form of this fraction has *a fortiori* a denominator not exceeding b; hence it belongs to $\mathfrak{F}_n$. Supposing, moreover, that O were not on any of the straight lines that join two of the elements of α, then five terms would come between a/b and $(a+1)/(b-1)$, so that $m > 5$, by (3). We may therefore say that either O does not lie in Λ_α, or it is in a straight line with two points of α.

Replacing, next, α by β, it follows as before that O is either not in the region Λ_β (Fig. 2), where

$$5y-x < 0, \qquad 2y-x > 3,$$

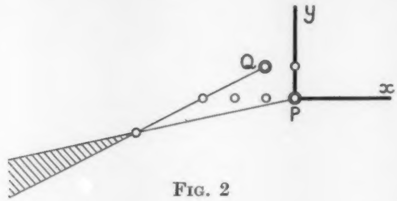

FIG. 2

or collinear with two points of β. A similar alternative can be derived from γ, the corresponding region Λ_γ (Fig. 3) being defined by

$$3y-x < 0, \qquad 3y-2x > 5,$$

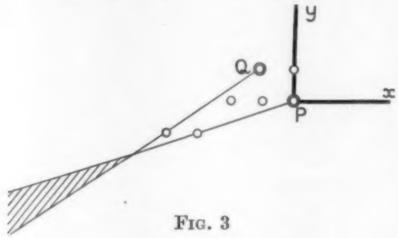

FIG. 3

and again from δ when Λ_δ (Fig. 4) is given by

$$2y-x < 0, \qquad y-x > 2.$$

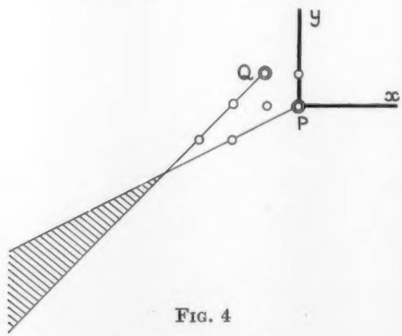

FIG. 4

We can sum up what we have deduced in the form: O is a lattice-point of Λ and (i) not in Λ_α, Λ_β, Λ_γ, Λ_δ, or (ii) on a line joining two points out of one of the sets α, β, γ, δ.

Consider the case (i). The regions Λ_α, Λ_β, Λ_γ, Λ_δ overlap and cover Λ, apart from a small area M (Fig. 5). We reckon in M the points on its boundary but not on the boundary of Λ. Then O may be any of the lattice-points of M, with the exception of such points as are separated from P by lattice-points; for a/b is irreducible, and so no lattice-point lies between O and P. As possible positions of O seven points remain (black dots in Fig. 5), and P represents any of the fractions

$$\tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \tfrac{1}{7},\ \tfrac{2}{5},\ \tfrac{2}{7},\ \tfrac{3}{7}. \tag{7}$$

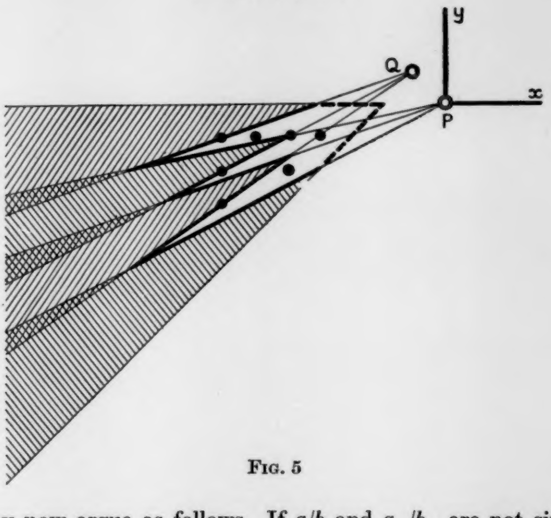

FIG. 5

We may now argue as follows. If a/b and a_m/b_m are not similarly ordered, then a_m/b_m is, by (2), a term of every $\mathfrak{F}_n$ that contains a/b. Furthermore, if a/b and a_m/b_m are mth neighbours in $\mathfrak{F}_n$, they are not nearer in Farey series of higher order. Take first $a/b = \frac{1}{4}$. In $\mathfrak{F}_4$ only $\frac{2}{3}$ is a fraction following $\frac{1}{4}$ such that the two are not similarly ordered; and $\frac{1}{4}$, $\frac{2}{3}$ are third neighbours in $\mathfrak{F}_4$, fifth neighbours in $\mathfrak{F}_5$ and $\mathfrak{F}_6$, and farther distant in $\mathfrak{F}_7$ and all later Farey series. Next let $a/b = \frac{1}{5}$. In $\mathfrak{F}_5$ the term $\frac{1}{5}$ and any of the five terms after are similarly ordered. Thus we have to do away with the possibility of P representing $\frac{1}{5}$. In that way we can go through the fractions (7).

The results will be listed along with those obtained from the alternative (ii) which remains to be examined.

There are ten lines each joining a pair out of α, β, γ, or δ, and penetrating into Λ. As an illustration, we suppose that O lies on the line $2y-x = 2$; in this case O might be any of the lattice-points inside Λ and not separated by lattice-points from P. Then P represents, in turn, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{5}{12}$,.... These fractions can be scrutinized as before. When O is beyond $(-14, -6)$, the fractions represented by A and by the points $(-1-2i, -i)$ $(i = 0, 1, 2, 3)$ come between a/b and $(a+1)/(b-1)$, and no two of those points can be collinear with O. Similar considerations apply to all lines joining two points of α, β, γ, δ.

Eventually we have the following table:

	Not similarly ordered		
	third	*fourth*	*fifth*
	neighbours in		
$\frac{1}{4}$, $\frac{2}{3}$	$\mathfrak{F}_4$		$\mathfrak{F}_5$, $\mathfrak{F}_6$
$\frac{2}{5}$, $\frac{3}{4}$		$\mathfrak{F}_5$, $\mathfrak{F}_6$	
$\frac{1}{6}$, $\frac{2}{5}$		$\mathfrak{F}_6$	$\mathfrak{F}_7$
$\frac{1}{8}$, $\frac{2}{7}$			$\mathfrak{F}_8$
$\frac{3}{8}$, $\frac{4}{7}$		$\mathfrak{F}_8$	
$\frac{2}{9}$, $\frac{3}{8}$		$\mathfrak{F}_9$	$\mathfrak{F}_{10}$
$\frac{4}{9}$, $\frac{5}{8}$			$\mathfrak{F}_9$, $\mathfrak{F}_{10}$
$\frac{3}{10}$, $\frac{4}{9}$			$\mathfrak{F}_{10}$
$\frac{5}{12}$, $\frac{6}{11}$			$\mathfrak{F}_{12}$

We thus obtain these minima of N:

$$N(1) = 1, \quad N(2) = 1, \quad N(3) = 5, \quad N(4) = 10, \quad N(5) = 13.$$

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